INTERSECTION THEORY ON SHIMURA SURFACES II

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ABSTRACT. This is the third of a series of papers relating intersections of special cycles on the integral model of a Shimura surface to Fourier coefficients of Hilbert modular forms. More precisely, we embed the Shimura curve over $\mathbb{Q}$ associated to a rational quaternion algebra into the Shimura surface associated to the base change of the quaternion algebra to a real quadratic field. After extending the associated moduli problems over $\mathbb{Z}$ we obtain an arithmetic threefold with a embedded arithmetic surface, which we view as a cycle of codimension one. We then construct a family, indexed by totally positive algebraic integers in the real quadratic field, of codimension two cycles (complex multiplication points) on the arithmetic threefold. The intersection multiplicities of the codimension two cycles with the fixed codimension one cycle are shown to agree with the Fourier coefficients of a (very particular) Hilbert modular form of weight 3/2. The results are higher dimensional variants of results of Kudla-Rapoport-Yang, which relate intersection multiplicities of special cycles on the integral model of a Shimura curve to Fourier coefficients of a modular form in two variables.

1. INTRODUCTION

Let $F \subset \mathbb{R}$ be a real quadratic field of discriminant d_F, and fix a $\mathbb{Z}$-basis $\{\varpi_1, \varpi_2\}$ of $\mathcal{O}_F$. Let B_0 be a quaternion division algebra over $\mathbb{Q}$ satisfying

(a) $B_0 \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$ (fix one such isomorphism once and for all),

(b) every prime divisor of $\mathrm{disc}(B_0)$ splits in F,

and set $B = B_0 \otimes_{\mathbb{Q}} F$. Choose a maximal order $\mathcal{O}_{B_0}$ of B_0 that is stable under the main involution $b \mapsto b^\iota$, and set $\mathcal{O}_B = \mathcal{O}_{B_0} \otimes_{\mathbb{Z}} \mathcal{O}_F$. The hypothesis that all primes ramified in B_0 split in F implies that $\mathcal{O}_B$ is a maximal order of B. We consider two functors from the category of $\mathbb{Z}$-schemes to the category of groupoids (recall that a groupoid is a category in which all arrows are isomorphisms). The first, $\mathcal{M}_0$, associates to a $\mathbb{Z}$-scheme S the category of abelian schemes over S of relative dimension two equipped with an action of $\mathcal{O}_{B_0}$. The second, $\mathcal{M}$, associates to a $\mathbb{Z}$-scheme S the category of abelian schemes over S of relative dimension four equipped with an action of $\mathcal{O}_B$ (more precise definitions of these moduli problems are in §2; for the purposes of this introduction we omit some of the moduli data, $e.g.$ polarizations). The moduli problems $\mathcal{M}_0$ and $\mathcal{M}$ are representable by projective, regular Deligne-Mumford stacks of relative dimensions one and two, respectively, over $\mathrm{Spec}(\mathbb{Z})$, whose complex fibers are well-known from the theory of Shimura varieties. To describe these complex fibers define algebraic groups $G_0 \subset G$ over $\mathbb{Q}$ by

$$G_0(A) = (B_0 \otimes_{\mathbb{Q}} A)^\times$$

and
$$G(A) = \{x \in (B \otimes_{\mathbb{Q}} A)^{\times} : \mathrm{Nm}(x) \in A^{\times}\}$$
for any $\mathbb{Q}$-algebra A. Here Nm is the reduced norm on $B^{\times}$. Define discrete subgroups $\Gamma_0 = \mathcal{O}_{B_0}^{\times} \subset G_0(\mathbb{R})$ and
$$\Gamma = \{x \in \mathcal{O}_B^{\times} : \mathrm{Nm}(x) \in \mathbb{Z}^{\times}\} \subset G(\mathbb{R}).$$
Let $X_0 = \mathbb{P}^1(\mathbb{C}) \smallsetminus \mathbb{P}^1(\mathbb{R})$, and set
$$X = (X_0^+ \times X_0^+) \cup (X_0^- \times X_0^-)$$
where X_0^+ and X_0^- are the connected components of X_0. If we identify $G_0(\mathbb{R})$ with $\mathrm{GL}_2(\mathbb{R})$, and identify $G(\mathbb{R})$ with a subgroup of $\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$, there are isomorphisms of complex orbifolds
$$\mathcal{M}_0(\mathbb{C}) \cong [\Gamma_0 \backslash X_0] \qquad \mathcal{M}(\mathbb{C}) \cong [\Gamma \backslash X].$$

For an abelian scheme A_0 over an arbitrary base scheme S there is an abelian scheme $A_0 \otimes \mathcal{O}_F$ whose functor of points satisfies $(A_0 \otimes \mathcal{O}_F)(T) \cong A_0(T) \otimes_{\mathbb{Z}} \mathcal{O}_F$ for any S-scheme T [12, §3.1]. There is a canonical closed immersion $\mathcal{M}_0 \to \mathcal{M}$, which is given on moduli by $A_0 \mapsto A_0 \otimes \mathcal{O}_F$, and which on complex fibers is induced by the diagonal embedding $X_0 \to X$. This closed immersion induces a linear functional (defined in §2)

$$(1.1) \qquad\qquad \widehat{\deg}_{\mathcal{M}_0} : \widehat{\mathrm{CH}}^2(\mathcal{M}) \to \mathbb{R}$$

on the codimension two Gillet-Soulé arithmetic Chow group of $\mathcal{M}$ (with rational coefficients) called the *arithmetic degree along* $\mathcal{M}_0$. In an earlier work [12] the author constructed an arithmetic cycle class

$$(1.2) \qquad\qquad \widehat{\mathcal{Y}}(\alpha, v) \in \widehat{\mathrm{CH}}^2(\mathcal{M})$$

depending on a totally positive $\alpha \in \mathcal{O}_F$ and a totally positive $v \in F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R} \times \mathbb{R}$. The construction of this class is based upon another moduli problem, $\mathcal{Y}(\alpha)$, which associates to a $\mathbb{Z}$-scheme S the category of abelian schemes over S equipped with an action of $\mathcal{O}_B$ and a commuting action of $\mathcal{O}_F[\sqrt{-\alpha}]$. Roughly speaking, $\mathcal{Y}(\alpha)$ is the moduli space of points on $\mathcal{M}$ with complex multiplication by the order $\mathcal{O}_F[\sqrt{-\alpha}]$. The evident forgetful map $\mathcal{Y}(\alpha) \to \mathcal{M}$ is finite and unramified, allowing one to view $\mathcal{Y}(\alpha)$ as a cycle on $\mathcal{M}$. The cycle

$$\mathcal{Y}(\alpha) \times_{\mathbb{Z}} \mathbb{Z}[1/\mathrm{disc}(B_0)] \to \mathcal{M} \times_{\mathbb{Z}} \mathbb{Z}[1/\mathrm{disc}(B_0)]$$

has codimension two, and may (depending on α) have nonreduced vertical components at primes that are nonsplit in F. At primes dividing $\mathrm{disc}(B_0)$, the cycle $\mathcal{Y}(\alpha)$ may have vertical components of codimension one in $\mathcal{M}$. One of the main results of [12] is the construction of certain natural replacements for these components of excess dimension, and we will briefly recall the essentials of this construction in §7. The codimension two cycle on $\mathcal{M}$ underlying the cycle class (1.2) is then $\mathcal{Y}(\alpha)$ with the modified vertical components at primes dividing $\mathrm{disc}(B_0)$. The totally positive parameter $v \in F \otimes_{\mathbb{Q}} \mathbb{R}$ is used in the construction of a Green current for this cycle.

As the value

$$(1.3) \qquad\qquad \widehat{\deg}_{\mathcal{M}_0} \widehat{\mathcal{Y}}(\alpha, v)$$

is essentially the intersection multiplicity of $\mathcal{Y}(\alpha)$ with $\mathcal{M}_0$ one would expect, following the general philosophy of Kudla [15, 16, 17] and the results of Kudla-Rapoport-Yang [20], that the arithmetic degree (1.3) should be related to Fourier coefficients

of the derivative of an Eisenstein series. The main result of [12] confirms that this is so, at least under the hypothesis that the field extension $F(\sqrt{-\alpha})/\mathbb{Q}$ is not bi-quadratic. This hypothesis ensures that the cycles $\mathcal{Y}(\alpha)$ and $\mathcal{M}_0$ are disjoint in the generic fiber of $\mathcal{M}$, so that the arithmetic degree along $\mathcal{M}_0$ is essentially the usual naive intersection multiplicity. In the present work we turn to the more difficult case in which $F(\sqrt{-\alpha})/\mathbb{Q}$ is biquadratic; thus we must compute intersection multiplicities of cycles that intersect improperly, in which case the definition of the arithmetic degree along $\mathcal{M}_0$ requires the use of Chow's moving lemma [23] on the generic fiber of $\mathcal{M}$. This complicates the picture considerably.

Before stating the main result, we describe the automorphic form to which (1.3) is to be related. To the quadratic space of trace zero elements of B_0, Kudla-Rapoport-Yang [20, (5.1.44)] attach an Eisenstein series $\mathcal{E}_2(\tau, s, B_0)$ of weight $3/2$ on the Siegel half-space of genus two $\mathfrak{h}_2$. This Eisenstein series satisfies a functional equation forcing $\mathcal{E}_2(\tau, 0, B_0) = 0$, and we denote by

$$\widehat{\phi}_2(\tau) = \mathcal{E}_2'(\tau, 0, B_0)$$

its derivative at $s = 0$. Let $\mathfrak{h}_1$ be the usual complex upper half-plane. The choice of $\mathbb{Z}$-basis $\{\varpi_1, \varpi_2\}$ of $\mathcal{O}_F$ determines an embedding $i_F : \mathfrak{h}_1 \times \mathfrak{h}_1 \to \mathfrak{h}_2$ by the rule

$$i_F(\tau_1, \tau_2) = R \begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix} {}^t R$$

where the matrix R is defined in (8.5). Pulling back $\widehat{\phi}_2(\tau)$ by this embedding results in a Hilbert modular form $i_F^* \widehat{\phi}_2(\tau_1, \tau_2)$ of weight $3/2$ for the real quadratic field F, having a Fourier expansion of the form

$$(1.4) \qquad i_F^* \widehat{\phi}_2(\tau_1, \tau_2) = \sum_{\alpha \in \mathcal{O}_F} c(\alpha, v) \cdot q^\alpha$$

in which $q^\alpha = e^{2\pi i \alpha \tau_1} e^{2\pi i \alpha^\sigma \tau_2}$, σ is the nontrivial Galois automorphism of $F/\mathbb{Q}$, and $v = (v_1, v_2)$ is the imaginary part of (τ_1, τ_2),

Our main theorem is the following.

Theorem A. *Suppose that $\alpha \in \mathcal{O}_F$ and $v \in F \otimes_\mathbb{Q} \mathbb{R}$ are both totally positive. If 2 splits in F and if $\alpha \mathcal{O}_F$ is relatively prime to the different of $F/\mathbb{Q}$ then*

$$\widehat{\deg}_{\mathcal{M}_0} \widehat{\mathcal{Y}}(\alpha, v) = c(\alpha, v).$$

Remark 1.1. If $F(\sqrt{-\alpha})/\mathbb{Q}$ is not biquadratic then one does not need to assume that 2 splits in F or that $\alpha \mathcal{O}_F$ is relatively prime to the different (see the main result of [12] or Theorem 8.2 below) and presumably the result is true if these hypotheses are omitted altogether. Both hypotheses are inherited from [11]. If p is nonsplit in F then the stack $\mathcal{Y}(\alpha)$ may have nonreduced vertical components in characteristic p, and if $p = 2$ the calculation of the multiplicities of these components in [11, Theorem C] breaks down in a serious way. To remove the assumption that $\alpha \mathcal{O}_F$ is prime to the different of $F/\mathbb{Q}$, one would have to extend the statement and proof of [11, Proposition 5.1.1] to include the case of $c_0 > 0$. Again, this probably requires some new ideas.

As for the proof of Theorem A, we begin by noting that $c(\alpha, v)$ has already been computed by Kudla-Rapoport-Yang [20]. Let

$$\mathrm{Sym}_2(\mathbb{Z})^\vee = \left\{ \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.$$

For each $T \in \mathrm{Sym}_2(\mathbb{Z})^\vee$ and positive definite symmetric $\mathbf{v} \in M_2(\mathbb{R})$, Kudla-Rapoport-Yang construct an arithmetic cycle class

$$\widehat{\mathcal{Z}}(T, \mathbf{v}) \in \widehat{\mathrm{CH}}_\mathbb{R}^2(\mathcal{M}_0)$$

(the arithmetic Chow group with real coefficients), and relate the Fourier coefficients of the genus two Siegel modular form $\widehat{\phi}_2(\tau)$ to the image of this class under the isomorphism

$$\widehat{\deg} : \widehat{\mathrm{CH}}_\mathbb{R}^2(\mathcal{M}_0) \to \mathbb{R}$$

of [20, §2.4]. More precisely, they prove that the Fourier expansion of $\widehat{\phi}_2(\tau)$ is

$$\widehat{\phi}_2(\tau) = \sum_{T \in \mathrm{Sym}_2(\mathbb{Z})^\vee} \widehat{\deg}\, \widehat{\mathcal{Z}}(T, \mathbf{v}) \cdot q^T$$

where $\mathbf{v}$ is the imaginary part of $\tau \in \mathfrak{h}_2$, and $q^T = e^{2\pi i \cdot \mathrm{Tr}(T\tau)}$. From this it is an easy exercise (see [12, Lemma 5.2.1]) to determine the Fourier coefficients of the pullback $\widehat{\phi}_2(\tau_1, \tau_2)$: for any $\alpha \in \mathcal{O}_F$ and any totally positive $v \in F \otimes_\mathbb{Q} \mathbb{R}$

$$c(\alpha, v) = \sum_{T \in \Sigma(\alpha)} \widehat{\deg}\, \widehat{\mathcal{Z}}(T, \mathbf{v}),$$

in which $\mathbf{v}$ and $v = (v_1, v_2) \in \mathbb{R} \times \mathbb{R}$ are related by (8.5) and

$$\Sigma(\alpha) = \left\{ \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in \mathrm{Sym}_2(\mathbb{Z})^\vee : \alpha = a\varpi_1^2 + b\varpi_1\varpi_2 + c\varpi_2^2 \right\}.$$

This leaves us with the problem of computing (1.3) for totally positive α and comparing with the values of $\widehat{\deg}\, \widehat{\mathcal{Z}}(T, \mathbf{v})$ (which are known by [20, Chapter 6]) in order to prove

$$(1.5) \qquad\qquad \widehat{\deg}_{\mathcal{M}_0} \widehat{\mathcal{Y}}(\alpha, v) = \sum_{T \in \Sigma(\alpha)} \widehat{\deg}\, \widehat{\mathcal{Z}}(T, \mathbf{v}),$$

from which Theorem A follows immediately. It is the calculation of the left hand side of (1.5) which occupies the entirety of this paper, culminating in Theorem 8.2.

In the calculation of (1.3) one encounters several obstacles. As noted earlier, the cycle $\mathcal{Y}(\alpha)$ on $\mathcal{M}$ may have components in codimension one, and so must be modified in order to obtain the cycle class (1.2). This is carried out in [12], and the construction of the modified components will be quickly recalled in §7. In order to compute (1.3) one must decompose the cycle $\mathcal{Y}(\alpha)$ component-by-component, and treat irreducible components in different ways depending on whether they meet $\mathcal{M}_0$ properly or improperly. If $\mathcal{D}$ is a component of $\mathcal{Y}(\alpha)$ that meets $\mathcal{M}_0$ improperly, then $\mathcal{D}$ is contained in $\mathcal{M}_0$. We attach an arithmetic cycle class

$$\widehat{\mathcal{D}}(v) \in \widehat{\mathrm{CH}}^2(\mathcal{M})$$

to $\mathcal{D}$ and prove an *arithmetic adjunction formula* (Theorem 5.6), which computes the arithmetic degree of $\widehat{\mathcal{D}}(v)$ along $\mathcal{M}_0$ in terms of data intrinsic to the divisor $\mathcal{D}$ on $\mathcal{M}_0$ (as opposed to data involving the relative positions of $\mathcal{D}$ and $\mathcal{M}_0$ in the larger threefold $\mathcal{M}$). While the method of derivation of the arithmetic adjunction formula should apply to the general problem of computing the arithmetic intersection of a codimension one cycle and a codimension two cycle meeting improperly in an arithmetic threefold, the final formula is rather specific to the case at hand, as it

makes use of the moduli interpretation of $i : \mathcal{M}_0 \to \mathcal{M}$. More precisely, an essential ingredient in the proof of Theorem 5.6 is the isomorphism of line bundles

$$i^*\omega \cong \omega_0 \otimes \omega_0$$

on $\mathcal{M}_0$, in which ω_0 and ω are the canonical bundles on $\mathcal{M}_0$ and $\mathcal{M}$, respectively. This isomorphism is proved in §4 by using the Kodaira-Spencer isomorphism to give moduli-theoretic interpretations of the canonical bundles.

This leaves the problem of computing the intersection of the remaining components of $\mathcal{Y}(\alpha)$ (those that are not contained in $\mathcal{M}_0$) against $\mathcal{M}_0$. For these components the arithmetic degree along $\mathcal{M}_0$ can be computed as a sum of local intersections at each prime, plus an archimedean contribution. Primes not dividing $\mathrm{disc}(B_0)$ are treated in §6 using formal deformation theory, while primes dividing $\mathrm{disc}(B_0)$ are treated in §7 using the Čerednik-Drinfeld uniformization of the formal completion of $\mathcal{M}_{/\mathbb{Z}_p}$ along its special fiber. Primes that are nonsplit in F cause the most difficulty, largely because of the presence of nonreduced vertical components in $\mathcal{Y}(\alpha)_{/\mathbb{Z}_p}$, whose multiplicities must be determined. As this calculation is very long and technical, it has been relegated to a separate article [11], which contains the bulk of the deformation theory calculations at primes nonsplit in F. It is Proposition 6.6 and its corollary Proposition 6.7 that make use of the calculations of [11]. Finally, the calculations of §5, §6, and §7 are combined in §8 to yield the final result Theorem 8.2, from which Theorem A follows.

As explained in the introduction to [12], Theorem A is one of the major steps toward the larger goal of proving a Gross-Zagier type theorem for Shimura surfaces, extending [20, Corollary 1.0.7] from Shimura curves to Shimura surfaces. The next step is to find a good definition of the class (1.2) for all $\alpha \in \mathcal{O}_F$ (not just for α totally positive). For $\alpha \neq 0$ but not totally positive the definition is straightforward: the cycle $\mathcal{Y}(\alpha)$ is empty, but the construction of the Green current $\Xi(\alpha, v)$ of §3, and the proof of (1.5) should pose no new difficulties. For $\alpha = 0$ the definition of (1.2) is more subtle, and would follow [20, §3.5] or [20, §6.5]. Roughly speaking, when $\alpha = 0$ the class (1.2) should be defined by viewing the metrized Hodge bundle of §4 as an element of $\widehat{\mathrm{CH}}^1(\mathcal{M})$, and taking its self-intersection. Once (1.2) has been defined for all α, the next step is to form the generating series

$$(1.6) \qquad \widehat{\theta}(\tau_1, \tau_2) = \sum_{\alpha \in \mathcal{O}_F} \widehat{\mathcal{Y}}(\alpha, v) \cdot q^\alpha \in \widehat{\mathrm{CH}}^2(\mathcal{M})[[q]].$$

The extension of Theorem A to all $\alpha \in \mathcal{O}_F$ would then prove the equality of power series

$$(1.7) \qquad \widehat{\mathrm{deg}}_{\mathcal{M}_0} \widehat{\theta}(\tau_1, \tau_2) = i_F^* \widehat{\phi}_2(\tau_1, \tau_2).$$

The next step is the most challenging. One would like to know, by analogy with [20, Theorem A], that the generating series (1.6) is a vector-valued, nonholomorphic, Hilbert modular form of weight $3/2$. If this is the case, then given a weight $3/2$ Hilbert modular cuspform f, the Petersson inner product of f with (1.6) defines a class

$$\widehat{\Theta}(f) \in \widehat{\mathrm{CH}}^2(\mathcal{M}).$$

If we denote by $L(f, s, B_0)$ the Petersson inner product of f with $i_F^* \mathcal{E}_2(\tau_1, \tau_2, s, B_0)$, the derivative $L'(f, 0, B_0)$ is equal to the Petersson inner product of f with $\widehat{\phi}_2$, and

(1.7) implies the Gross-Zagier style formula

$$\widehat{\deg}_{\mathcal{M}_0} \widehat{\Theta}(f) = L'(f, 0, B_0).$$

The most serious obstacle between Theorem A and this goal is the modularity of the generating series (1.6).

1.2. **Notation.** Denote by $\mathfrak{D}_F$ the different of $F/\mathbb{Q}$. Fix an embedding $F \hookrightarrow \mathbb{R}$, and let σ be the nontrivial Galois automorphism of $F/\mathbb{Q}$. Extend the ring homomorphism $F \to \mathbb{R} \times \mathbb{R}$ defined by $\alpha \otimes 1 \mapsto (\alpha, \alpha^\sigma)$ to an isomorphism of $\mathbb{R}$-algebras $F \otimes_\mathbb{Q} \mathbb{R} \cong \mathbb{R} \times \mathbb{R}$, denoted $v \mapsto (v_1, v_2)$.

Fix a positive involution $b \mapsto b^*$ of B_0 that leaves $\mathcal{O}_{B_0}$ stable and has the form $b^* = s^{-1} b^\iota s$ for some $s \in \mathcal{O}_{B_0}$ with $s^2 = -\mathrm{disc}(B_0)$. Extend $b \mapsto b^*$ to an involution of B that is trivial on F. If L is an algebraically closed field, $\mathcal{X}$ is any algebraic stack, and $P \in \mathcal{X}(L)$, denote by $\mathrm{Aut}_\mathcal{X}(P)$ the automorphism group of P in the category $\mathcal{X}(L)$.

2. Quaternionic Shimura varieties

In this section we recall some of the basic definitions and notation of [12], concerning abelian schemes with quaternionic multiplication, their moduli spaces, and the arithmetic Chow groups of those spaces.

By a *QM abelian surface* (QM is short for *quaternionic multiplication*) over a scheme S we mean a pair $\mathbf{A}_0 = (A_0, i_0)$ consisting of an abelian scheme $A_0 \to S$ of relative dimension two and an action $i_0 : \mathcal{O}_{B_0} \to \mathrm{End}(A_0)$ satisfying the Kottwitz condition of [12, §3.1]. A *principal polarization* of $\mathbf{A}_0$ is a principal polarization $\lambda_0 : A_0 \to A_0^\vee$ of the underlying abelian scheme that satisfies $\lambda_0 \circ i_0(b^*) = i_0(b)^\vee \circ \lambda_0$ for all $b \in \mathcal{O}_{B_0}$. By a *QM abelian fourfold* over a scheme S we mean a pair $\mathbf{A} = (A, i)$ consisting of an abelian scheme $A \to S$ of relative dimension four and an action $i : \mathcal{O}_B \to \mathrm{End}(A)$ satisfying the Kottwitz condition. A $\mathfrak{D}_F^{-1}$-*polarization* of $\mathbf{A}$ is a polarization $\lambda : A \to A^\vee$ of the underlying abelian fourfold that satisfies $\lambda \circ i(b^*) = i(b)^\vee \circ \lambda$ for all $b \in \mathcal{O}_B$ and whose kernel is $A[\mathfrak{D}_F]$. Such a λ determines an isomorphism $A \otimes_{\mathcal{O}_F} \mathfrak{D}_F^{-1} \to A^\vee$. By an endomorphism of $\mathbf{A}_0$ or $\mathbf{A}$ we mean an endomorphism of the underlying abelian scheme that commutes with the quaternionic action.

Let $\mathcal{M}_0$ be the Deligne-Mumford (DM) stack of principally polarized QM abelian surfaces over schemes, let $\mathcal{M}$ be the DM stack of $\mathfrak{D}_F^{-1}$-polarized QM abelian fourfolds over schemes, and let $i : \mathcal{M}_0 \to \mathcal{M}$ be the closed immersion defined by the functor

$$(\mathbf{A}_0, \lambda_0) \mapsto (\mathbf{A}_0, \lambda_0) \otimes \mathcal{O}_F = (\mathbf{A}_0 \otimes \mathcal{O}_F, \lambda_0 \otimes \mathcal{O}_F)$$

as in [12, §3.1]. The DM stacks $\mathcal{M}_0$ and $\mathcal{M}$ are regular of dimensions two and three, respectively, and are flat and projective over $\mathrm{Spec}(\mathbb{Z})$. If we abbreviate $D = \mathrm{disc}(B_0)$ then $\mathcal{M}_0$ is smooth over $\mathbb{Z}[1/D]$, and $\mathcal{M}$ is smooth over $\mathbb{Z}[1/(Dd_F)]$. For references to proofs of these properties, see [12, §3.1]. Briefly, for representability of $\mathcal{M}$ by a quasi-projective stack use [10, Chapters 6 and 7]; for properties of $\mathcal{M}$ over $\mathbb{Z}[(Dd_F)^{-1}]$ use [2, Exposé III], for properties at primes dividing d_F repeat

the arguments of [5], and for primes dividing D use the Cerednick-Drinfeld uni-
formization described in [12, §4]; projectivity follows from quasi-projectivity and
the valuative criterion of properness using [2, Exposé III.6]. It follows from [21,
§6.3.2] that i is a regular immersion and hence that $\mathcal{M}_0$ is an effective Cartier
divisor on $\mathcal{M}$.

If $(\mathbf{A}_0, \lambda_0)$ is a principally polarized QM abelian surface over a connected base
scheme S, then λ_0 determines a Rosati involution $\tau \mapsto \tau^\dagger$ on the $\mathbb{Q}$-algebra $\mathrm{End}^0(\mathbf{A}_0)$
of B_0-linear quasi-endomorphisms of $\mathbf{A}_0$. The *Rosati trace* on $\mathrm{End}^0(\mathbf{A}_0)$ is defined
by $\mathrm{Tr}(\tau) = \tau + \tau^\dagger$, and an endomorphism of $\mathbf{A}_0$ of Rosati trace zero is called a
special endomorphism. By [12, Lemma 3.1.2] the $\mathbb{Q}$-algebra $\mathrm{End}^0(\mathbf{A}_0)$ is either $\mathbb{Q}$,
a quadratic imaginary field, or a definite quaternion algebra. As the Rosati invo-
lution is positive, it must be (in the three cases respectively) the identity, complex
conjugation, or the main involution (in the case of a definite quaternion algebra,
this follows from Albert's classification of division algebras over $\mathbb{Q}$ with a positive
involution [22, Chapter 21]). In particular the Rosati trace agrees with the reduced
trace of [22, Chapter 19], and our definition of special endomorphism agrees with
the definition used in [20]. The $\mathbb{Z}$-module of special endomorphisms of $(\mathbf{A}_0, \lambda_0)$ is
equipped with the symmetric $\mathbb{Z}$-valued bilinear form $[\tau_1, \tau_2] = -\mathrm{Tr}(\tau_1 \tau_2)$ and its
associated quadratic form $Q_0(\tau) = -\tau^2$.

Similarly, if $(\mathbf{A}, \lambda)$ is a principally polarized QM abelian fourfold over S then
the F-algebra $\mathrm{End}^0(\mathbf{A})$ of B-linear quasi-endomorphisms of A comes equipped with
the F-linear Rosati involution $\tau \mapsto \tau^\dagger$ determined by λ, and the endomorphisms of
$\mathbf{A}$ of Rosati trace zero are again called *special endomorphisms*. The $\mathcal{O}_F$-module of
special endomorphisms of $(\mathbf{A}, \lambda)$ has a symmetric bilinear form $[\tau_1, \tau_2] = -\mathrm{Tr}(\tau_1 \tau_2)$
and an associated quadratic form $Q(\tau) = -\tau^2$, each of which is $\mathcal{O}_F$-valued.

For each nonzero $t \in \mathbb{Z}$ define, following [20, §3.4], $\mathcal{Z}(t)$ to be the DM stack of
triples $(\mathbf{A}_0, \lambda_0, s_0)$ in which $(\mathbf{A}_0, \lambda_0)$ is a principally polarized QM abelian surface
over a scheme and $s_0 \in \mathrm{End}(\mathbf{A}_0)$ is a special endomorphism that satisfies $Q_0(s_0) = t$. We view $\mathcal{Z}(t)$ also as a codimension one cycle on $\mathcal{M}_0$. This means that every
irreducible component of $\mathcal{Z}(t)$ is viewed as an irreducible cycle of $\mathcal{M}_0$ via the
forgetful morphism, and is weighted according to the length of the strictly Henselian
local ring at its generic point. By [20, Proposition 3.4.5] the cycle $\mathcal{Z}(t)$ has no
vertical components except possibly at primes dividing $\mathrm{disc}(B_0)$. As a cycle we
decompose $\mathcal{Z}(t) = \mathcal{Z}^{\mathrm{hor}}(t) + \mathcal{Z}^{\mathrm{ver}}(t)$ into its horizontal and vertical parts, and then
further decompose

$$\mathcal{Z}^{\mathrm{ver}}(t) = \sum_{p | \mathrm{disc}(B_0)} \mathcal{Z}^{\mathrm{ver}}(t)_p.$$

As in [20, §3.6], for each nonzero $T \in \mathrm{Sym}_2(\mathbb{Z})^\vee$ let $\mathcal{Z}(T)$ be the DM stack of
quadruples $(\mathbf{A}_0, \lambda_0, s_1, s_2)$, in which $(\mathbf{A}_0, \lambda_0)$ is as above and $s_1, s_2 \in \mathrm{End}(\mathbf{A}_0)$ are
special endomorphisms that satisfy

$$(2.1) \qquad \frac{1}{2} \begin{pmatrix} [s_1, s_1] & [s_1, s_2] \\ [s_1, s_2] & [s_2, s_2] \end{pmatrix} = T.$$

If $\det(T) \neq 0$ then, by [20, Theorem 3.6.1], $\mathcal{Z}(T)$ is either empty or all of its points
have residue field of the same characteristic $p \neq 0$. If this characteristic p does not
divide $\mathrm{disc}(B_0)$ then $\mathcal{Z}(T)$ is of dimension zero, while if p does divide $\mathrm{disc}(B_0)$ then
$\mathcal{Z}(T)$ may have vertical components of dimension one. If $T \neq 0$ but $\det(T) = 0$,

then there is a $t \in \mathbb{Z}$ for which $\mathcal{Z}(T) \cong \mathcal{Z}(t)$. See [20, Lemma 6.4.1] or (6.8) below. In any case one has dim $\mathcal{Z}(T) \leq 1$.

For each nonzero $\alpha \in \mathcal{O}_F$ let $\mathcal{Y}(\alpha)$ be the DM stack of triples $(\mathbf{A}, \lambda, t_\alpha)$, in which $(\mathbf{A}, \lambda)$ is a $\mathfrak{D}_F^{-1}$-polarized QM abelian fourfold over a scheme and t_α is a special endomorphism of $\mathbf{A}$ satisfying $Q(t_\alpha) = \alpha$. Let $\phi : \mathcal{Y}(\alpha) \to \mathcal{M}$ be the functor that forgets the data t_α, and define $\mathcal{Y}_0(\alpha) = \mathcal{Y}(\alpha) \times_{\mathcal{M}} \mathcal{M}_0$, so that there is a cartesian diagram

$$
\begin{array}{ccc}
\mathcal{Y}_0(\alpha) & \xrightarrow{\phi_0} & \mathcal{M}_0 \\
{\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} \\
\mathcal{Y}(\alpha) & \xrightarrow{\phi} & \mathcal{M},
\end{array}
$$

in which both vertical arrows are closed immersions and both horizontal arrows are proper and quasi-finite, hence finite. As in [12, §3.1], there is a canonical decomposition

$$
(2.2) \qquad \mathcal{Y}_0(\alpha) \cong \bigsqcup_{T \in \Sigma(\alpha)} \mathcal{Z}(T)
$$

defined as follows. An object of the category $\mathcal{Y}_0(\alpha)$ consists of an object $(\mathbf{A}, \lambda, t_\alpha)$ of $\mathcal{Y}(\alpha)$, an object $(\mathbf{A}_0, \lambda_0)$ of $\mathcal{M}_0$, and an isomorphism $(\mathbf{A}, \lambda) \cong (\mathbf{A}_0, \lambda_0) \otimes \mathcal{O}_F$. This isomorphisms determines an isomorphism

$$
\mathrm{End}(\mathbf{A}) \cong \mathrm{End}(\mathbf{A}_0) \otimes \mathcal{O}_F
$$

which allows us to write $t_\alpha = s_1 \varpi_1 + s_2 \varpi_2$ for some special endomorphisms $s_1, s_2 \in \mathrm{End}(\mathbf{A}_0)$. The condition $Q(t_\alpha) = \alpha$ is equivalent to the condition that the matrix (2.1) lies in $\Sigma(\alpha)$, and the isomorphism (2.2) takes the above data to the quadruple $(\mathbf{A}_0, \lambda_0, s_1, s_2)$.

Remark 2.1. If $P \in \mathcal{M}(L)$ with L an algebraically closed field, define

$$
e_P = |\mathrm{Aut}_{\mathcal{M}(L)}(P)|.
$$

If $P \in \mathcal{M}_0(L)$ then we will routinely confuse P with its image in $\mathcal{M}(L)$. As [12, Lemma 3.1.1] implies that the automorphism group of P in $\mathcal{M}_0(L)$ is isomorphic to the automorphism group of P in $\mathcal{M}(L)$, we also have

$$
e_P = |\mathrm{Aut}_{\mathcal{M}_0(L)}(P)|.
$$

Similarly, if $P \in \mathcal{Z}(T)(L)$ (respectively $P \in \mathcal{Z}(t)(L)$) then e_P denotes the size of the automorphism group of P in $\mathcal{M}_0(L)$, where P is regarded as an object of this latter category via the obvious forgetful map $\mathcal{Z}(T) \to \mathcal{M}_0$ (respectively $\mathcal{Z}(t) \to \mathcal{M}_0$).

The remainder of this section is devoted to the careful construction of the linear functional (1.1). If $\mathcal{X}$ is either $\mathcal{M}_0$ or $\mathcal{M}$, we let $\widehat{Z}^k(\mathcal{X})$ be the $\mathbb{Q}$-vector space of pairs $(\mathcal{D}, \Xi)$, in which $\mathcal{D}$ is a codimension k cycle on $\mathcal{X}$ with rational coefficients, and Ξ is an equivalence class of Green currents for $\mathcal{D}$. The codimension k arithmetic Chow group $\widehat{\mathrm{CH}}^k(\mathcal{X})$ of $\mathcal{X}$, as defined by Gillet-Soulé [1, 7, 26], is the quotient of $\widehat{Z}^k(\mathcal{X})$ by the subspace spanned by pairs of the form

$$
\widehat{\mathrm{div}}(f) = \big(\mathrm{div}(f), [-\log|f|^2]\big)
$$

for f a rational function on an integral substack of $\mathcal{X}$ of codimension $k - 1$. Any class $\widehat{\mathcal{D}} \in \widehat{\mathrm{CH}}^2(\mathcal{M})$ may be represented by a pair $(\mathcal{D}, \Xi_{\mathcal{D}})$ in which $\mathcal{D}$ has support

disjoint from $\mathcal{M}_0$ in the generic fiber (by first expressing the generic fiber $\mathcal{M}_{/\mathbb{Q}}$ as a the quotient of a $\mathbb{Q}$-scheme M by the action of a finite group H, applying the Moving Lemma over $\mathbb{Q}$ proved in [23], and then averaging over H). Thus to define $\widehat{\deg}_{\mathcal{M}_0}\widehat{\mathcal{D}}$ we may assume that $\mathcal{D}$ is irreducible and is disjoint from $\mathcal{M}_0$ in the generic fiber of $\mathcal{M}$. The arithmetic degree along $\mathcal{M}_0$ is then defined as a sum of local contributions, which we now describe.

Fix a prime p and an isomorphism of stacks $\mathcal{M}_{/\mathbb{Z}_p} \cong [H\backslash M]$ with M a $\mathbb{Z}_p$-scheme and H a finite group of automorphisms of M (for example by imposing prime-to-p level structure on the moduli problem defining the stack $\mathcal{M}$). Set

$$(2.3) \qquad M_0 = \mathcal{M}_0 \times_{\mathcal{M}} M.$$

If D is an irreducible cycle of codimension two on M that is not contained in M_0, define the *Serre intersection multiplicity* at p

$$(2.4) \qquad I_p(D, M_0) = \sum_{x \in M_0(\mathbb{F}_p^{\mathrm{alg}})} \sum_{\ell \geq 0} (-1)^\ell \cdot \mathrm{length}_{\mathcal{O}_{M_0,x}} \mathrm{Tor}_\ell^{\mathcal{O}_{M,x}}(\mathcal{O}_{D,x}, \mathcal{O}_{M_0,x})$$

where we view both $\mathcal{O}_D$ and $\mathcal{O}_{M_0}$ as coherent $\mathcal{O}_M$-modules. In fact only the $\ell = 0$ term contributes to the right hand side: as D is integral of dimension one, the local ring of $\mathcal{O}_D$ at any point of D is a Cohen-Macaulay local ring, and hence the stalk $\mathcal{O}_{D,x}$ at any $x \in M(\mathbb{F}_p^{\mathrm{alg}})$ is a Cohen-Macaulay $\mathcal{O}_{M,x}$-module [25, p. 63]. The regularity of M_0 implies that the stalk $\mathcal{O}_{M_0,x}$ is also Cohen-Macaulay as an $\mathcal{O}_{M,x}$-module, and hence

$$(2.5) \qquad \mathrm{Tor}_\ell^{\mathcal{O}_{M,x}}(\mathcal{O}_{D,x}, \mathcal{O}_{M_0,x}) = 0$$

for $\ell > 0$ by [25, p. 111].

The definition (2.4) can be extended to cycles supported in characteristic p, including those that meet M_0 improperly. If $\mathcal{F}_0$ is a coherent $\mathcal{O}_{M_{0/\mathbb{F}_p}}$-module, define the Euler characteristic

$$\chi(\mathcal{F}_0) = \sum_{k \geq 0} (-1)^k \dim_{\mathbb{F}_p} H^k(M_{0/\mathbb{F}_p}, \mathcal{F}_0).$$

If the sheaf $\mathcal{F}_0$ is supported in dimension zero then

$$(2.6) \qquad \chi(\mathcal{F}_0) = \sum_{x \in M_0(\mathbb{F}_p^{\mathrm{alg}})} \mathrm{length}_{\mathcal{O}_{M_0,x}} \mathcal{F}_{0,x}.$$

For any irreducible vertical cycle D of codimension two on M, the coherent $\mathcal{O}_M$-module $\mathrm{Tor}_\ell^{\mathcal{O}_{M,x}}(\mathcal{O}_{D,x}, \mathcal{O}_{M_0,x})$ is annihilated by p and by the ideal sheaf of the closed subscheme $M_0 \to M$, and hence may be viewed as a coherent $\mathcal{O}_{M_{0/\mathbb{F}_p}}$-module. Thus we may define

$$(2.7) \qquad I_p(D, M_0) = \sum_{\ell \geq 0} (-1)^\ell \cdot \chi(\mathrm{Tor}_\ell^{\mathcal{O}_M}(\mathcal{O}_D, \mathcal{O}_{M_0})).$$

If D is both vertical and not contained in M_0 then one sees using (2.6) that the two definitions (2.4) and (2.7) of $I_p(D, M_0)$ agree. By extending $I_p(D, M_0)$ linearly in the first variable, we then define $I_p(D, M_0)$ for any codimension two cycle D on M whose support is disjoint from M_0 in the generic fiber. If f is a rational function on any irreducible component of $M_{/\mathbb{F}_p}$ and D is the associated Weil divisor on $M_{/\mathbb{F}_p}$, viewed as a vertical cycle on M of codimension two, then one can show that $I_p(D, M_0) = 0$. By [6, Lemma 4.2], any codimension two cycle $\mathcal{D}$ on $\mathcal{M}$ with support

disjoint from $\mathcal{M}_0$ in the generic fiber determines an H-invariant codimension two cycle D on M that is disjoint from M_0 in the generic fiber. Thus we may define

$$I_p(\mathcal{D}, \mathcal{M}_0) = \frac{1}{|H|} I_p(D, M_0).$$

Now consider the situation at the infinite place. Choose an isomorphism of stacks $\mathcal{M}_{/\mathbb{Q}} \cong [H \backslash M]$ with M a $\mathbb{Q}$-scheme and H a finite group of automorphisms of M. Again define M_0 by (2.3). Suppose that D is any codimension two cycle on M that is disjoint from M_0, and that Ξ_D is a Green current for D in the sense of [7, §1.2]. We give two definitions of $I_\infty(\Xi_D, M_0)$. The first definition uses the methods of [7, §1.3]. We say that two currents Ξ and Ξ' on M (or on M_0) are *equivalent* if there are smooth currents u and v such that

$$\Xi' = \Xi + \partial u + \overline{\partial} v.$$

One may replace Ξ_D by an equivalent current Ξ'_D that is a Green form of logarithmic type for D. In particular, as the support of D is assumed to be disjoint from M_0, Ξ'_D is represented by a smooth $(1,1)$-form in a complex neighborhood of M_0, and the pullback $i^*\Xi'_D$ is a differential form of top degree on the smooth manifold $M_0(\mathbb{C})$. Define

$$I_\infty(\Xi_D, M_0) = \frac{1}{2} \int_{M_0(\mathbb{C})} i^*\Xi'_D.$$

The second definition uses the methods of [7, §2.1.5]. Briefly, one can construct a family $\{\omega^\epsilon\}_{\epsilon > 0}$ of smooth $(1,1)$-forms on $M(\mathbb{C})$ that converge, as $\epsilon \to 0$, to the delta current δ_{M_0} on M. One then defines

$$I_\infty(\Xi_D, M_0) = \lim_{\epsilon \to 0} \frac{1}{2} \int_{M(\mathbb{C})} \omega^\epsilon \wedge \Xi_D$$

where the integral on the right is understood to mean evaluation of the current $\omega^\epsilon \wedge \Xi_D$ at the constant function 1. One checks that this definition agrees with the first definition using [7, §2.2.12]. Now suppose $\mathcal{D}$ is a codimension two cycle on $\mathcal{M}$ and let D be the associated H-invariant cycle on M as in the previous paragraph. A Green current $\Xi_\mathcal{D}$ for $\mathcal{D}$ is defined to be an H-invariant Green current Ξ_D for D. If $\mathcal{D}$ has support disjoint from $\mathcal{M}_0$ in the generic fiber and $\Xi_\mathcal{D}$ is a Green current for $\mathcal{D}$, we define

$$I_\infty(\Xi_\mathcal{D}, \mathcal{M}_0) = \frac{1}{|H|} I_\infty(\Xi_D, M_0).$$

The *arithmetic degree along* $\mathcal{M}_0$ of $\widehat{\mathcal{D}}$

$$\widehat{\deg}_{\mathcal{M}_0} \widehat{\mathcal{D}} = I_\infty(\Xi, \mathcal{M}_0) + \sum_{p \text{ prime}} I_p(\mathcal{D}, \mathcal{M}_0) \log(p)$$

does not depend on the choice of representative $(\mathcal{D}, \Xi_\mathcal{D})$, and defines the desired linear functional

$$(2.8) \qquad\qquad \widehat{\deg}_{\mathcal{M}_0} : \widehat{\mathrm{CH}}^2(\mathcal{M}) \to \mathbb{R}.$$

One proves that the arithmetic degree along $\mathcal{M}_0$ does not depend of the choice of $(\mathcal{D}, \Xi_\mathcal{D})$ representing $\widehat{\mathcal{D}}$ by showing that the definition given above agrees with the definition found in [12, §2.3].

3. Complex uniformization

Fix a totally positive $\alpha \in \mathcal{O}_F$ and abbreviate $\mathcal{Y} = \mathcal{Y}(\alpha)$. In this section we review the well-known complex uniformizations of $\mathcal{M}_0(\mathbb{C})$ and $\mathcal{M}(\mathbb{C})$, and construct a Green current for the 0-cycle $\mathcal{Y}(\mathbb{C})$ on $\mathcal{M}(\mathbb{C})$.

Choose an isomorphism of stacks $\mathcal{M}_{/\mathbb{Q}} \cong [H\backslash M]$ with M a $\mathbb{Q}$-scheme and H a finite group of automorphisms of M, and abbreviate

$$Y = \mathcal{Y} \times_{\mathcal{M}} M.$$

Recalling the forgetful map $\phi : Y \to M$, we attach to Y the 0-cycle on M

$$(3.1) \qquad C_{\mathbb{Q}} = \sum_{y \in Y} \text{length}_{\mathcal{O}_{Y,y}}(\mathcal{O}_{Y,y}) \cdot \phi(y)$$

which, using [6, Lemma 4.2], descends to a codimension two cycle $\mathcal{C}_{\mathbb{Q}}$ on $\mathcal{M}_{/\mathbb{Q}}$ independent of the choice of presentation $M \to \mathcal{M}_{/\mathbb{Q}}$. There is a unique decomposition

$$\mathcal{C}_{\mathbb{Q}} = \mathcal{C}_{\mathbb{Q}}^{\bullet} + \mathcal{C}_{\mathbb{Q}}^{\bullet\bullet}$$

of cycles on $\mathcal{M}_{/\mathbb{Q}}$ such that $\mathcal{C}_{\mathbb{Q}}^{\bullet\bullet}$ is supported on $\mathcal{M}_{0/\mathbb{Q}}$ and $\mathcal{C}_{\mathbb{Q}}^{\bullet}$ has support disjoint from $\mathcal{M}_{0/\mathbb{Q}}$. We will construct Green currents for the cycles $\mathcal{C}_{\mathbb{Q}}^{\bullet}$ and $\mathcal{C}_{\mathbb{Q}}^{\bullet\bullet}$.

Remark 3.1. In (3.1) we in fact have $\text{length}_{\mathcal{O}_{Y,y}}(\mathcal{O}_{Y,y}) = 1$ for each $y \in Y$. Indeed, $\mathcal{Y}_{/\mathbb{Q}}$ is étale over $\text{Spec}(\mathbb{Q})$ by [12, Lemma 3.1.3], and in particular Y is a disjoint union of spectra of number fields.

Let X_0 and X be as in the introduction. The obvious inclusion $X \to X_0 \times X_0$ is denoted $x \mapsto (x_1, x_2)$ and $\pi_i : X \to X_0$ denotes the function $\pi_i(x) = x_i$. Let

$$\mu_0 = y^{-2} \cdot dx \wedge dy$$

be the usual hyperbolic volume form on X_0. For any positive $u \in \mathbb{R}$, Kudla [20, §7.3] has constructed a symmetric Green function $g_u^0(z_1, z_2)$ for the diagonal on $X_0 \times X_0$, and a smooth symmetric function $\phi_u^0(z_1, z_2)$ on $X_0 \times X_0$. These functions have the property that for any fixed $x_0 \in X_0$, the smooth function in the variable $z_0 \in X_0 \smallsetminus \{x_0\}$

$$\mathbf{g}_0(x_0, u)(z_0) = g_u^0(x_0, z_0)$$

and the smooth $(1,1)$-form on X_0

$$\Phi_0(x_0, u)(z_0) = \phi_u^0(x_0, z_0)\mu_0(z_0)$$

satisfy the Green equation

$$dd^c \mathbf{g}_0(x_0, u) + \delta_{x_0} = \Phi_0(x_0, u)$$

of $(1,1)$-currents on X_0. For a point $x \in X^{\pm}$ and a pair $v = (v_1, v_2)$ of positive real numbers the functions

$$\mathbf{g}_1(x, v) = \pi_1^* \mathbf{g}_0(x_1, v_1) \qquad \mathbf{g}_2(x, v) = \pi_2^* \mathbf{g}_0(x_2, v_2)$$

are Green functions for the divisors $\{x_1\} \times X_0^{\pm}$ and $X_0^{\pm} \times \{x_2\}$ on $X^{\pm}$. Extend these functions by 0 to $X^{\mp}$. If we define $(1,1)$-forms

$$\Phi_1(x, v) = \pi_1^* \Phi_0(x_1, v_1) \qquad \Phi_2(x, v) = \pi_2^* \Phi_0(x_2, v_2)$$

on $X^{\pm}$ and extend by 0 to $X^{\mp}$, then the star product defined by [7, §2.1]

$$\begin{aligned}\mathbf{g}(x,v) &= \mathbf{g}_1(x,v) * \mathbf{g}_2(x,v) \\ &= \mathbf{g}_1(x,v) \wedge \delta_{X_0^{\pm} \times \{x_2\}} + \mathbf{g}_2(x,v) \wedge \Phi_1(x,v) \\ &= \mathbf{g}_2(x,v) \wedge \delta_{\{x_1\} \times X_0^{\pm}} + \mathbf{g}_1(x,v) \wedge \Phi_2(x,v)\end{aligned}$$

is a $(1,1)$-current on X satisfying the Green equation

$$dd^c \mathbf{g}(x,v) + \delta_x = \Phi_1(x,v) \wedge \Phi_2(x,v).$$

Our fixed isomorphism $B_0 \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$ determines an isomorphism $G_0(\mathbb{R}) \cong \mathrm{GL}_2(\mathbb{R})$, and hence determines an action of $G_0(\mathbb{R})$ on X_0. The induced isomorphism $B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R}) \times M_2(\mathbb{R})$ then determines an isomorphism

$$G(\mathbb{R}) \cong \{(g_1, g_2) \in \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R}) : \det(g_1) = \det(g_2)\},$$

and hence an action of $G(\mathbb{R})$ on X. The inclusion $G(\mathbb{R}) \to G_0(\mathbb{R}) \times G_0(\mathbb{R})$ is denoted $\gamma \mapsto (\gamma_1, \gamma_2)$. By Shimura's theory there are orbifold presentations

(3.2) $$\mathcal{M}_0(\mathbb{C}) \cong [\Gamma_0 \backslash X_0] \qquad \mathcal{M}(\mathbb{C}) \cong [\Gamma \backslash X],$$

and the morphism $\mathcal{M}_0(\mathbb{C}) \to \mathcal{M}(\mathbb{C})$ of §2 is induced by the diagonal inclusion $X_0 \to X$.

Following [12, §3.2] the fibers of the universal QM abelian surface on $[\Gamma_0 \backslash X_0]$ can be described as follows. For each $z_0 \in X_0$ define an isomorphism of real vector spaces

$$\rho_{0,z_0} : B_0 \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{C}^2 \qquad \rho_{0,z_0}(A) = A \cdot \begin{bmatrix} z_0 \\ 1 \end{bmatrix}$$

and set $\Lambda_{0,z_0} = \rho_{0,z_0}(\mathcal{O}_{B_0})$. Then $\mathbf{A}_{0,z_0} = \mathbb{C}^2/\Lambda_{0,z_0}$ is a QM abelian surface, and the perfect alternating pairing $\psi_0 : \mathcal{O}_{B_0} \times \mathcal{O}_{B_0} \to \mathbb{Z}$ defined in [12, §3.1] determines a pairing ψ_{0,z_0} on $\Lambda_{0,z}$ with the property that one of $\pm \psi_{0,z_0}$ (depending on the connected component of X_0 containing z_0) is a Riemann form. Thus $\mathbf{A}_{0,z_0}$ comes equipped with a principal polarization λ_{0,z_0}, and $(\mathbf{A}_{0,z_0}, \lambda_{0,z_0})$ is a QM abelian surface that depends only on the Γ_0-orbit of z_0. Similarly, for each $z \in X$ we write (z_1, z_2) for the corresponding point of $X_0 \times X_0$ and define an isomorphism

$$\rho_z : B \otimes_{\mathbb{Q}} \mathbb{R} \cong (B_0 \otimes_{\mathbb{Q}} \mathbb{R}) \times (B_0 \otimes_{\mathbb{Q}} \mathbb{R}) \cong \mathbb{C}^2 \times \mathbb{C}^2$$

by $\rho_z = \rho_{0,z_1} \times \rho_{0,z_2}$. Set $\Lambda_z = \rho_z(\mathcal{O}_B)$. Extend ψ_0 $\mathcal{O}_F$-linearly to an $\mathcal{O}_F$-valued pairing on $\mathcal{O}_B \cong \mathcal{O}_{B_0} \otimes_{\mathbb{Z}} \mathcal{O}_F$, and define $\psi = \mathrm{Tr}_{F/\mathbb{Q}} \circ \psi_0$. As above, ψ determines a pairing ψ_z on Λ_z, and one of $\pm \psi_z$ is a Riemann form for Λ_z. The resulting polarization λ_z of $\mathbf{A}_z = (\mathbb{C}^2 \times \mathbb{C}^2)/\Lambda_z$ determines a $\mathfrak{D}_F^{-1}$-polarized QM abelian fourfold $(\mathbf{A}_z, \lambda_z)$ that depends only on the Γ-orbit of z.

Let V_0 and V denote the trace zero elements of B_0 and B, respectively, with $G_0(\mathbb{Q})$ and $G(\mathbb{Q})$ acting on V_0 and V by conjugation. We identify

$$V \otimes_{\mathbb{Q}} \mathbb{R} \cong (V_0 \otimes_{\mathbb{Q}} \mathbb{R}) \times (V_0 \otimes_{\mathbb{Q}} \mathbb{R})$$

and write $\tau \mapsto (\tau_1, \tau_2)$ for the isomorphism. The F-vector space V is endowed with the $G(\mathbb{Q})$-invariant F-valued quadratic form $Q(\tau) = -\tau^2$, and V_0 is endowed with the $G_0(\mathbb{Q})$-invariant $\mathbb{Q}$-valued quadratic form Q_0 defined by the same formula. Each $\tau_0 \in V_0 \otimes_{\mathbb{Q}} \mathbb{R}$ with $Q_0(\tau)$ positive, viewed as an element of $G_0(\mathbb{R})$, acts on X_0 with two fixed points

$$x_0^{\pm}(\tau_0) \in X_0^{\pm},$$

and the $(0,0)$-current on X_0 defined by

$$\xi_0(\tau_0) = \mathbf{g}_0\big(x_0^+(\tau_0), Q_0(\tau_0)\big) + \mathbf{g}_0\big(x_0^-(\tau_0), Q_0(\tau_0)\big)$$

is a Green current for the 0-cycle $x_0^+(\tau_0) + x_0^-(\tau_0)$. Given $\tau \in V \otimes_{\mathbb{Q}} \mathbb{R}$ with $Q(\tau)$ totally positive, the fixed points of τ acting on X are

$$x^+(\tau) = (x_0^+(\tau_1), x_0^+(\tau_2)) \qquad x^-(\tau) = (x_0^-(\tau_1), x_0^-(\tau_2)),$$

and the $(1,1)$-current on X defined by

$$\xi(\tau) = \mathbf{g}\big(x^+(\tau), Q(\tau)\big) + \mathbf{g}\big(x^-(\tau), Q(\tau)\big)$$

is a Green current for the 0-cycle $x^+(\tau) + x^-(\tau)$. Set $L = V \cap \mathcal{O}_B$. For a totally positive $v \in F \otimes_{\mathbb{Q}} \mathbb{R}$ the current

$$(3.3) \qquad \Xi(\alpha, v) = \sum_{\substack{\tau \in L \\ Q(\tau) = \alpha}} \xi(v^{1/2}\tau) = \sum_{\substack{\tau \in L \\ Q(\tau) = \alpha}} \big(\mathbf{g}(x^+(\tau), \alpha v) + \mathbf{g}(x^-(\tau), \alpha v)\big)$$

is Γ-invariant and so descends to a $(1,1)$-current on the orbifold $\mathcal{M}(\mathbb{C})$ which we denote in the same way. Decompose $L = L^{\mathrm{sing}} \sqcup L^{\mathrm{nsing}}$ in which L^{sing} (resp. L^{nsing}) is the subset consisting of those τ for which the vectors $\tau_1, \tau_2 \in V_0 \otimes_{\mathbb{Q}} \mathbb{R}$ are linearly dependent (resp. linearly independent). Note that

$$(3.4) \qquad\qquad\qquad \tau \in L^{\mathrm{sing}} \iff x^{\pm}(\tau) \in X_0.$$

We now decompose $L = L^{\bullet} \sqcup L^{\bullet\bullet}$ in which

$$L^{\bullet\bullet} = \Gamma L^{\mathrm{sing}} \qquad L^{\bullet} = L \smallsetminus L^{\bullet\bullet}$$

and define $(1,1)$-currents $\Xi^{\bullet}(\alpha, v)$ and $\Xi^{\bullet\bullet}(\alpha, v)$ on X exactly as in (3.3), but with L replaced by $L^{\bullet}$ and $L^{\bullet\bullet}$, respectively. Both currents are invariant under the action of Γ and so define currents on the orbifold $\mathcal{M}(\mathbb{C})$.

Proposition 3.2. *The currents* $\Xi(\alpha, v)$, $\Xi^{\bullet}(\alpha, v)$, *and* $\Xi^{\bullet\bullet}(\alpha, v)$ *are Green currents for the* 0-*cycles* $\mathcal{C}_{\mathbb{Q}}$, $\mathcal{C}_{\mathbb{Q}}^{\bullet}$, *and* $\mathcal{C}_{\mathbb{Q}}^{\bullet\bullet}$ *on* $\mathcal{M}$, *respectively.*

Proof. It suffices to prove any two of the three claims. As in [12, §3.2] there is an isomorphism of zero dimensional orbifolds

$$\mathcal{Y}(\alpha)(\mathbb{C}) \cong \Big[\Gamma \backslash \bigsqcup_{\substack{\tau \in L \\ Q(\tau) = \alpha}} \{x^+(\tau), x^-(\tau)\}\Big].$$

This shows that the pullback of the cycle $\mathcal{C}_{\mathbb{Q}}$ to X is equal to the formal sum

$$\sum_{\substack{\tau \in L \\ Q(\tau) = \alpha}} (x^+(\tau) + x^-(\tau)),$$

which has $\Xi(\alpha, v)$ as a Green current. The pullback of $\mathcal{C}_{\mathbb{Q}}^{\bullet\bullet}$ to X consists of that portion of the above sum whose support is contained in the Γ-orbit of X_0. This is the formal sum

$$\sum_{\substack{\tau \in L^{\bullet\bullet} \\ Q(\tau) = \alpha}} (x^+(\tau) + x^-(\tau)),$$

which has $\Xi^{\bullet\bullet}(\alpha, v)$ as a Green current. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. The Hodge bundle

In this section we define the hodge line bundles ω_0^{Hdg} and ω^{Hdg} on $\mathcal{M}_0$ and $\mathcal{M}$, respectively, and prove that these line bundles are isomorphic to the canonical bundles of these stacks. We then construct explicit metrics on these bundles, and prove that the restriction of ω^{Hdg} to $\mathcal{M}_0$ is isomorphic, as a metrized line bundle, to $\omega_0^{\mathrm{Hdg}} \otimes \omega_0^{\mathrm{Hdg}}$. This will be a key ingredient in the arithmetic adjunction formula proved in §5.

Abbreviate $\delta_F = \varpi_1 \varpi_2^\sigma - \varpi_2 \varpi_1^\sigma$, where $\{\varpi_1, \varpi_2\}$ is our fixed $\mathbb{Z}$-basis of $\mathcal{O}_F$. Note that $-\delta_F \delta_F^\sigma = \delta_F^2 = d_F$, and that $\delta_F \mathcal{O}_F = \mathfrak{D}_F$. For an abelian scheme $A \to S$ with identity section $e : S \to A$, the *co-Lie algebra* of A is the locally free $\mathcal{O}_S$-module
$$\mathrm{coLie}(A/S) = e^* \Omega^1_{A/S} \cong \mathrm{Hom}_{\mathcal{O}_S}(\mathrm{Lie}(A/S), \mathcal{O}_S)$$
of rank equal to the relative dimension of A. Define the *Hodge bundle* on $\mathcal{M}_0$ by
$$\omega_0^{\mathrm{Hdg}} = \wedge^2 \mathrm{coLie}(\mathbf{A}_0^{\mathrm{univ}}/\mathcal{M}_0)$$
where the exterior product is taken in the category of $\mathcal{O}_{\mathcal{M}_0}$-modules, and $\mathbf{A}_0^{\mathrm{univ}}$ is the universal QM abelian surface over $\mathcal{M}_0$. Define the Hodge bundle on $\mathcal{M}$ by
$$\omega^{\mathrm{Hdg}} = \wedge^2 \wedge^2_{\mathcal{O}_F} \mathrm{coLie}(\mathbf{A}^{\mathrm{univ}}/\mathcal{M})$$
where the $\wedge^2$ means exterior square in the category of $\mathcal{O}_\mathcal{M}$-modules and $\wedge^2_{\mathcal{O}_F}$ means exterior square in the category of $\mathcal{O}_\mathcal{M} \otimes_\mathbb{Z} \mathcal{O}_F$-modules. The Hodge bundle is essentially the determinant bundle of $\mathrm{coLie}(\mathbf{A}^{\mathrm{univ}}/\mathcal{M})$. Indeed, if $\mathcal{L}$ is any $\mathcal{O}_{\mathcal{M}_{/\mathbb{C}}} \otimes_\mathbb{Q} F$-module that is locally free of rank two, the splitting $\mathbb{C} \otimes_\mathbb{Q} F \cong \mathbb{C} \times \mathbb{C}$ induces a decomposition $\mathcal{L} \cong \mathcal{L}^{(1)} \oplus \mathcal{L}^{(2)}$. There is then an isomorphism of line bundles
$$\wedge^2 \wedge^2_{\mathcal{O}_F} \mathcal{L} \to \wedge^4 \mathcal{L}$$
on $\mathcal{M}_{/\mathbb{C}}$ determined by
$$(s_1 \wedge t_1) \wedge (s_2 \wedge t_2) \mapsto s_1 \wedge t_1 \wedge s_2 \wedge t_2$$
for local sections s_1, t_1 of $\mathcal{L}^{(1)}$ and s_2, t_2 of $\mathcal{L}^{(2)}$. Taking $\mathcal{L} = \mathrm{coLie}(\mathbf{A}^{\mathrm{univ}}/\mathcal{M})$ shows
$$(4.1) \qquad\qquad \omega_{/\mathbb{C}}^{\mathrm{Hdg}} \cong \wedge^4 \mathrm{coLie}(\mathbf{A}^{\mathrm{univ}}/\mathcal{M})_{/\mathbb{C}}.$$

Lemma 4.1. *Recalling the closed immersion* $i : \mathcal{M}_0 \to \mathcal{M}$ *of §2, there is an isomorphism of invertible $\mathcal{O}_{\mathcal{M}_0}$-modules*
$$i^* \omega^{\mathrm{Hdg}} \cong \omega_0^{\mathrm{Hdg}} \otimes \omega_0^{\mathrm{Hdg}}.$$

Proof. Using the canonical isomorphism
$$i^* \mathrm{Lie}(\mathbf{A}^{\mathrm{univ}}/\mathcal{M}) \cong \mathrm{Lie}(\mathbf{A}_0^{\mathrm{univ}}/\mathcal{M}_0) \otimes_\mathbb{Z} \mathcal{O}_F$$
of $\mathcal{O}_{\mathcal{M}_0} \otimes_\mathbb{Z} \mathcal{O}_F$-modules, we deduce that
$$i^* \mathrm{coLie}(\mathbf{A}^{\mathrm{univ}}/\mathcal{M}) \cong \mathrm{coLie}(\mathbf{A}_0^{\mathrm{univ}}/\mathcal{M}_0) \otimes_\mathbb{Z} \mathfrak{D}_F^{-1}.$$
If $\mathcal{L}_0$ is an $\mathcal{O}_{\mathcal{M}_0}$-module that is locally free of rank two, there is an isomorphism
$$\wedge^2_{\mathcal{O}_F}(\mathcal{L}_0 \otimes_\mathbb{Z} \mathfrak{D}_F^{-1}) \to (\wedge^2 \mathcal{L}_0) \otimes_\mathbb{Z} \mathcal{O}_F$$
defined by
$$(s \otimes \delta) \wedge (s' \otimes \delta') \mapsto (s \wedge s') \otimes (\delta \delta' \cdot d_F)$$
and an isomorphism
$$\wedge^2 \big((\wedge^2 \mathcal{L}_0) \otimes_\mathbb{Z} \mathcal{O}_F\big) \to (\wedge^2 \mathcal{L}_0) \otimes (\wedge^2 \mathcal{L}_0) \otimes (\wedge^2 \mathcal{O}_F)$$

defined by
$$\big((s \wedge t) \otimes x\big) \wedge \big((s' \wedge t') \otimes x'\big) \mapsto (s \wedge t) \otimes (s' \wedge t') \otimes (x \wedge x').$$

Using the $\mathbb{Z}$-basis $\{\varpi_1, \varpi_2\}$ of $\mathcal{O}_F$ to identify $\wedge^2 \mathcal{O}_F \cong \mathbb{Z}$ via
$$(a\varpi_1 + b\varpi_2) \wedge (c\varpi_1 + d\varpi_2) \mapsto ad - bc$$

and applying the above isomorphisms with $\mathcal{L}_0 = \mathrm{coLie}(\mathbf{A}_0^{\mathrm{univ}}/\mathcal{M}_0)$, we find isomorphisms

$$
\begin{aligned}
i^* \omega^{\mathrm{Hdg}}
&\cong \wedge^2 \wedge^2_{\mathcal{O}_F} (\mathrm{coLie}(\mathbf{A}_0^{\mathrm{univ}}/\mathcal{M}_0) \otimes_{\mathbb{Z}} \mathfrak{D}_F^{-1}) \\
&\cong \wedge^2 \big((\wedge^2 \mathrm{coLie}(\mathbf{A}_0^{\mathrm{univ}}/\mathcal{M}_0)) \otimes_{\mathbb{Z}} \mathcal{O}_F\big) \\
&\cong \big(\wedge^2 \mathrm{coLie}(\mathbf{A}_0^{\mathrm{univ}}/\mathcal{M}_0)\big) \otimes \big(\wedge^2 \mathrm{coLie}(\mathbf{A}_0^{\mathrm{univ}}/\mathcal{M}_0)\big) \\
&\cong \omega_0^{\mathrm{Hdg}} \otimes \omega_0^{\mathrm{Hdg}}.
\end{aligned}
$$

$\square$

Let $\omega_0 = \omega_{\mathcal{M}_0/\mathbb{Z}}$ and $\omega = \omega_{\mathcal{M}/\mathbb{Z}}$ be the canonical bundles on $\mathcal{M}_0$ and $\mathcal{M}$, respectively. There is a canonical morphism of $\mathcal{O}_{\mathcal{M}_0}$-modules $\Omega^1_{\mathcal{M}_0} \to \omega_0$, which is an isomorphism when restricted to the smooth locus of $\mathcal{M}_0 \to \mathrm{Spec}(\mathbb{Z})$, and hence we may identify $\omega_{0/\mathbb{C}} \cong \Omega^1_{\mathcal{M}_{0/\mathbb{C}}}$. Similarly there is a canonical morphism of $\mathcal{O}_{\mathcal{M}}$-modules $\Omega^2_{\mathcal{M}} \to \omega$, which is an isomorphism over the smooth locus of $\mathcal{M} \to \mathrm{Spec}(\mathbb{Z})$, and so we may identify $\omega_{/\mathbb{C}} \cong \Omega^2_{\mathcal{M}_{/\mathbb{C}}}$.

Proposition 4.2. *There are isomorphisms*
$$\omega_0 \cong \omega_0^{\mathrm{Hdg}} \qquad \omega \cong \omega^{\mathrm{Hdg}}$$
of line bundles on $\mathcal{M}_0$ and $\mathcal{M}$, respectively.

Proof. The first isomorphism is [19, Proposition 3.2] (compare also with [14, §1.0]). We will give a slightly different construction, which is better suited to the calculations to be performed in the proof of Proposition 4.4. Suppose that $U \to \mathcal{M}_0$ is an étale morphism with U a scheme, and that U is smooth over $\mathrm{Spec}(\mathbb{Z})$. That is to say, U is an étale open subset of the smooth locus of $\mathcal{M}_0$. The morphism $U \to \mathcal{M}_0$ determines a principally polarized QM abelian surface (A_0, i_0, λ_0) over U, and the order $\mathcal{O}_{B_0}$ acts naturally on the right on each of the $\mathcal{O}_U$-modules $\mathrm{coLie}(A_0/U)$, $\mathrm{Lie}(A_0^\vee/U)$, and $H^1_{\mathrm{DR}}(A_0/U)$. Given an $\mathcal{O}_{B_0} \otimes_{\mathbb{Z}} \mathcal{O}_U$-linear map
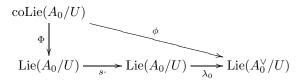$$\phi : \mathrm{coLie}(A_0/U) \to \mathrm{Lie}(A_0^\vee/U)$$

we will attach to ϕ a skew-symmetric $\mathcal{O}_U$-bilinear pairing Q_ϕ on $\mathrm{coLie}(A_0/U)$ in such a way that the construction $\phi \mapsto Q_\phi$ determines an isomorphism

$$(4.2) \quad \mathrm{Hom}_{\mathcal{O}_{B_0} \otimes_{\mathbb{Z}} \mathcal{O}_U}(\mathrm{coLie}(A_0/U), \mathrm{Lie}(A_0^\vee/U)) \cong \mathrm{Hom}_{\mathcal{O}_U}(\wedge^2 \mathrm{coLie}(A_0/U), \mathcal{O}_U).$$

Indeed, from the proof of [19, Proposition 3.2] one deduces the existence of a unique $\mathcal{O}_U$-linear map
$$\Phi : \mathrm{coLie}(A_0/U) \to \mathrm{Lie}(A_0/U)$$
satisfying $\Phi(x \cdot b) = b^\iota \cdot \Phi(x)$ for all $b \in \mathcal{O}_{B_0}$ and making the diagram

$$
\begin{array}{c}
\mathrm{coLie}(A_0/U) \\
{\scriptstyle \Phi} \downarrow {\scriptstyle \phi} \searrow \\
\mathrm{Lie}(A_0/U) \xrightarrow{\ s\cdot\ } \mathrm{Lie}(A_0/U) \xrightarrow{\ \lambda_0\ } \mathrm{Lie}(A_0^\vee/U)
\end{array}
$$

commute, and one shows that the pairing

$$Q_\phi(x, y) = \langle x, \Phi(y) \rangle$$

has the desired properties. Here $s \in \mathcal{O}_{B_0}$ is the trace free element chosen in the definition of the involution $b^* = s^{-1}b^\iota s$ on B_0, $b \mapsto b^\iota$ is the main involution, and the pairing $\langle \cdot, \cdot \rangle$ is the tautological pairing between $\mathrm{coLie}(A_0/U)$ and $\mathrm{Lie}(A_0/U)$.

Denote by $T_{U/\mathbb{Z}} = \mathrm{Hom}_{\mathcal{O}_U}(\Omega^1_{U/\mathbb{Z}}, \mathcal{O}_U)$ the tangent sheaf of U. For any local section D of $T_{U/\mathbb{Z}}$ the Gauss-Manin connection determines an $\mathcal{O}_{B_0}$-linear morphism of coherent $\mathcal{O}_U$-modules

$$\nabla(D) : H^1_{\mathrm{DR}}(A_0/U) \to H^1_{\mathrm{DR}}(A_0/U).$$

Combining this with the short exact sequence

$$0 \to \mathrm{coLie}(A_0/U) \to H^1_{\mathrm{DR}}(A_0/U) \to \mathrm{Lie}(A_0^\vee/U) \to 0$$

of Hodge theory yields the *Kodaira-Spencer isomorphism*

$$(4.3) \qquad\qquad T_{U/\mathbb{Z}} \to \mathrm{Hom}_{\mathcal{O}_{B_0} \otimes_{\mathbb{Z}} \mathcal{O}_U}(\mathrm{coLie}(A_0/U), \mathrm{Lie}(A_0^\vee/U))$$

defined by sending D to the composition

$$\mathrm{coLie}(A_0/U) \to H^1_{\mathrm{DR}}(A_0/U) \xrightarrow{\nabla(D)} H^1_{\mathrm{DR}}(A_0/U) \to \mathrm{Lie}(A_0^\vee/U).$$

More details on this construction can be found in [13, §1]. Composing the Kodaira-Spencer isomorphism with the isomorphism (4.2) and dualizing yields an isomorphism

$$\wedge^2 \mathrm{coLie}(A_0/U) \to \Omega^1_{U/\mathbb{Z}},$$

and thus over the smooth locus of $\mathcal{M}_0$ there is an isomorphism $\omega_0^{\mathrm{Hdg}} \cong \omega_0$ of line bundles. Using the regularity of $\mathcal{M}_0$ and the fact that the nonsmooth locus of $\mathcal{M}_0$ lies in codimension two, one shows that this isomorphism extends uniquely across all of $\mathcal{M}_0$.

Now suppose that $U \to \mathcal{M}$ is an étale open subset of the smooth locus of $\mathcal{M}$, and let (A, i, λ) be the corresponding $\mathfrak{D}_F^{-1}$-polarized QM abelian fourfold over U. We claim that, as above, there is an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_B \otimes_{\mathbb{Z}} \mathcal{O}_U}(\mathrm{coLie}(A/U), \mathrm{Lie}(A^\vee/U))$$
$$(4.4) \qquad\qquad \cong \mathrm{Hom}_{\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_U}(\wedge^2_{\mathcal{O}_F} \mathrm{coLie}(A/U), \mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_U),$$

which we will denote by $\phi \mapsto Q_\phi$. The construction of Q_ϕ is essentially the same as that considered earlier. There is a unique perfect $\mathcal{O}_F$-bilinear pairing

$$\mathrm{coLie}(A/U) \otimes_{\mathcal{O}_U} \mathrm{Lie}(A/U) \to \mathfrak{D}_F^{-1} \otimes_{\mathbb{Z}} \mathcal{O}_U$$

such that the composition

$$\mathrm{coLie}(A/U) \otimes_{\mathcal{O}_U} \mathrm{Lie}(A/U) \to \mathfrak{D}_F^{-1} \otimes_{\mathbb{Z}} \mathcal{O}_U \xrightarrow{\mathrm{Tr}_{F/\mathbb{Q}} \otimes \mathrm{id}} \mathcal{O}_U$$

is the tautological pairing. This pairing defines the first arrow in the $\mathcal{O}_F$-bilinear pairing

$$\mathrm{coLie}(A/U) \otimes_{\mathcal{O}_U} (\mathrm{Lie}(A/U) \otimes_{\mathcal{O}_F} \mathfrak{D}_F^{-1}) \to \mathfrak{D}_F^{-2} \otimes_{\mathbb{Z}} \mathcal{O}_U \xrightarrow{d_F \otimes \mathrm{id}} \mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_U,$$

which we denote by $\langle \cdot, \cdot \rangle$. View the polarization λ as an isomorphism

$$A \otimes_{\mathcal{O}_F} \mathfrak{D}_F^{-1} \cong A^\vee.$$

For each $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathcal{O}_U$-linear

$$\phi : \mathrm{coLie}(A/U) \to \mathrm{Lie}(A^\vee/U)$$

there is a unique $\mathcal{O}_U$-linear map

$$\Phi : \mathrm{coLie}(A/U) \to \mathrm{Lie}(A/U) \otimes_{\mathcal{O}_F} \mathfrak{D}_F^{-1}$$

satisfying $\Phi(x \cdot b) = b^\iota \cdot \Phi(x)$ for all $b \in \mathcal{O}_B$ and making the diagram

$$
\begin{array}{ccc}
\mathrm{coLie}(A/U) & & \\
{\scriptstyle \Phi}\downarrow & \searrow^{\phi} & \\
\mathrm{Lie}(A/U) \otimes_{\mathcal{O}_F} \mathfrak{D}_F^{-1} \xrightarrow{\ s\cdot\ } \mathrm{Lie}(A/U) \otimes_{\mathcal{O}_F} \mathfrak{D}_F^{-1} \xrightarrow{\ \lambda\ } \mathrm{Lie}(A^\vee/U)
\end{array}
$$

commute, and the pairing

$$Q_\phi(x,y) = \langle x, \Phi(y) \rangle$$

has the desired properties. The $\mathbb{Z}$-module homomorphism

$$\mathcal{O}_F \xrightarrow{\ \delta_F^{-1}\ } \mathfrak{D}_F^{-1} \xrightarrow{\ \mathrm{Tr}_{F/\mathbb{Q}}\ } \mathbb{Z}$$

induces an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_U}(\wedge^2_{\mathcal{O}_F} \mathrm{coLie}(A/U), \mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_U) \to \mathrm{Hom}_{\mathcal{O}_U}(\wedge^2_{\mathcal{O}_F} \mathrm{coLie}(A/U), \mathcal{O}_U),$$

which when composed with (4.4) and the Kodaira-Spencer isomorphism

$$T_{U/\mathbb{Z}} \to \mathrm{Hom}_{\mathcal{O}_B \otimes_{\mathbb{Z}} \mathcal{O}_U}(\mathrm{coLie}(A/U), \mathrm{Lie}(A^\vee/U))$$

yields an isomorphism

$$\wedge^2_{\mathcal{O}_F} \mathrm{coLie}(A/U) \cong \Omega^1_{U/\mathbb{Z}}.$$

Thus over the smooth locus of $\mathcal{M}$ there is an isomorphism $\omega^{\mathrm{Hdg}} \cong \omega$ which, again using the regularity of $\mathcal{M}$ and the fact that the nonsmooth locus of $\mathcal{M}$ lies in codimension two, extends uniquely across all of $\mathcal{M}$. $\square$

Remark 4.3. In the sequel we freely identify ω_0 with ω_0^{Hdg} and ω with ω^{Hdg} using the isomorphisms of Proposition 4.2.

Let σ_0 and τ_0 denote the coordinate functions on $\mathbb{C}^2$. For each $z_0 \in X_0$ the holomorphic 2-form $d\sigma_0 \wedge d\tau_0$ on $\mathbb{C}^2$ defines a holomorphic 2-form on the QM abelian surface $\mathbf{A}_{0,z_0}$ constructed in §3, hence an element of the stalk at z_0 of the pullback of $\omega_{0/\mathbb{C}}^{\mathrm{Hdg}}$ to X_0. As z_0 varies

$$\epsilon_0 = d\sigma_0 \wedge d\tau_0$$

defines a nonvanishing global section of the pullback of $\omega_{0/\mathbb{C}}$ to X_0. Similarly, if $\sigma_1, \tau_1, \sigma_2, \tau_2$ are the coordinate functions on $\mathbb{C}^2 \times \mathbb{C}^2$, then $d\sigma_i \wedge d\tau_i$ for $i = 1, 2$ defines a section of the pullback of $\wedge^2_{\mathcal{O}_F} \mathrm{coLie}(\mathbf{A}^{\mathrm{univ}}/\mathcal{M})_{/\mathbb{C}}$ to X, and hence

$$\epsilon = \delta_F^{-1} \cdot (d\sigma_1 \wedge d\tau_1) \wedge (d\sigma_2 \wedge d\tau_2)$$

defines a section of the pullback of $\omega_{/\mathbb{C}}^{\mathrm{Hdg}}$ to X. Tracing through the above constructions shows that the isomorphism of Lemma 4.1 satisfies

$$(4.5) \qquad\qquad i^* \epsilon \mapsto d_F \cdot \epsilon_0 \otimes \epsilon_0.$$

We now metrize ω_0^{Hdg} and ω^{Hdg}. For a point $z_0 \in \mathcal{M}_0(\mathbb{C})$ and a vector u_0 in the fiber of ω_0^{Hdg} at z_0, define

$$(4.6) \qquad \|u_0\|_{z_0}^2 = \frac{1}{2^4 \pi^3 e^{\gamma_{\mathrm{Euler}}}} \left| \int_{\mathbf{A}_{0,z_0}^{\mathrm{univ}}(\mathbb{C})} u_0 \wedge \overline{u}_0 \right|$$

where $\gamma_{\mathrm{Euler}} = 0.5772\ldots$ is Euler's constant. Denote by $\widehat{\omega}_0$ the line bundle ω_0^{Hdg} equipped with the above metric, and note that our $\widehat{\omega}_0$ is precisely the metrized Hodge bundle constructed by Kudla-Rapoport-Yang in [19, Definition 3.4]. As in [19, (3.15)] the explicit construction of $\mathbf{A}_{z_0}$ given in §3, together with the easy calculation

$$\mathrm{Vol}(M_2(\mathbb{R})/\mathcal{O}_{B_0}) = \mathrm{disc}(B_0)$$

(the volume is with respect to the Haar measure on $M_2(\mathbb{R})$ normalized so that $\mathrm{Vol}(M_2(\mathbb{R})/M_2(\mathbb{Z})) = 1$) shows that after pulling back ω_0^{Hdg} to X_0

$$(4.7) \qquad \|\epsilon_0\|_{z_0}^2 = \frac{1}{2^2 \pi^3 e^{\gamma_{\mathrm{Euler}}}} \mathrm{Im}(z_0)^2 \mathrm{disc}(B_0).$$

We metrize ω^{Hdg} in a similar way. For a point $z \in \mathcal{M}(\mathbb{C})$ and a vector u in the fiber of ω^{Hdg} at z, we use the isomorphism (4.1) to view u as a holomorphic 4-form on the QM abelian fourfold $\mathbf{A}_z^{\mathrm{univ}}$ and define

$$(4.8) \qquad \|u\|_z^2 = \frac{1}{2^8 \pi^6 e^{2\gamma_{\mathrm{Euler}}}} \left| \int_{\mathbf{A}_z^{\mathrm{univ}}(\mathbb{C})} u \wedge \overline{u} \right|.$$

Pulling back ω^{Hdg} to X, and using the volume calculation

$$\mathrm{Vol}\big((M_2(\mathbb{R}) \times M_2(\mathbb{R}))/\mathcal{O}_B\big) = \mathrm{disc}(B_0)^2 \cdot d_F^2$$

and the construction of $\mathbf{A}_z$ of §3, we then compute

$$(4.9) \qquad \|\epsilon\|_z^2 = \frac{d_F^2}{2^4 \pi^6 e^{2\gamma_{\mathrm{Euler}}}} \mathrm{Im}(z_1)^2 \mathrm{Im}(z_2)^2 \mathrm{disc}(B_0)^2.$$

Comparing (4.5), (4.7), and (4.9) we find that the isomorphism of Lemma 4.1 preserves the metrics defined above. That is to say, the isomorphism of Lemma 4.1 induces an isomorphism of metrized line bundles

$$(4.10) \qquad\qquad i^* \widehat{\omega} \cong \widehat{\omega}_0 \otimes \widehat{\omega}_0.$$

Proposition 4.4. *The metrics (4.6) and (4.8) on ω_0^{Hdg} and ω^{Hdg} induce metrics on the sheaves of top degree holomorphic differential forms $\Omega_{X_0}^1$ and Ω_X^2, and these metrics are determined by the formulas*

$$\|dz_0\|^2 = \frac{1}{\pi e^{\gamma_{\mathrm{Euler}}}} \cdot \mathrm{Im}(z_0)^2 \cdot \mathrm{disc}(B_0)$$

and

$$\|dz_1 \wedge dz_2\|^2 = \frac{d_F}{\pi^2 e^{2\gamma_{\mathrm{Euler}}}} \cdot \mathrm{Im}(z_1)^2 \mathrm{Im}(z_2)^2 \cdot \mathrm{disc}(B_0)^2,$$

respectively.

Proof. Return to the notation of §3, and in particular recall the family of principally polarized QM abelian surfaces $(A_{0,z_0}, i_{0,z_0}, \lambda_{0,z_0})$ parametrized by $z_0 = x_0 + iy_0 \in X_0$. By Hodge theory there is an isomorphism of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{coLie}(A_{0,z_0}/\mathbb{C}) & \longrightarrow & H^1_{\mathrm{DR}}(A_{0,z_0}/\mathbb{C}) & \longrightarrow & H^1(A_{0,z_0}, \mathcal{O}_{A_{0,z_0}}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^{1,0}(A_{0,z_0}/\mathbb{C}) & \longrightarrow & H^1_{\mathrm{DR}}(A_{0,z_0}/\mathbb{C}) & \longrightarrow & H^{0,1}(A_{0,z_0}/\mathbb{C}) & \longrightarrow & 0,
\end{array}
$$

and the cohomology of the exponential sequence

$$
0 \to 2\pi i \mathbb{Z} \to \mathcal{O}_{A_{0,z_0}} \xrightarrow{f \mapsto e^f} \mathcal{O}^\times_{A_{0,z_0}} \to 0
$$

induces the first isomorphism in

$$
\mathrm{Lie}(A^\vee_{0,z_0}/\mathbb{C}) \cong H^1(A_{0,z_0}, \mathcal{O}_{A_{0,z_0}}) \cong H^{0,1}(A_{0,z_0}/\mathbb{C}).
$$

Recall that we have fixed an isomorphism $B_0 \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$ and defined an isomorphism of $\mathbb{R}$-vector spaces $\rho_{0,z_0} : M_2(\mathbb{R}) \to \mathbb{C}^2$ by

$$
\rho_{0,z_0}(A) = A \cdot \begin{bmatrix} z_0 \\ 1 \end{bmatrix}.
$$

If we give $M_2(\mathbb{R})$ the complex structure under which multiplication by i is equal to right multiplication by

$$
J_{z_0} = \frac{1}{y_0} \begin{pmatrix} y_0 & x_0 \\ & 1 \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & -x_0 \\ & y_0 \end{pmatrix},
$$

then ρ_{0,z_0} is an isomorphism of complex vector spaces. By definition of A_{0,z_0} there are isomorphisms of smooth manifolds

$$
(4.11) \qquad M_2(\mathbb{R})/\mathcal{O}_{B_0} \xrightarrow{\rho_{0,z_0}} \mathbb{C}^2/\rho_{0,z_0}(\mathcal{O}_{B_0}) \cong A_{0,z_0}.
$$

Let σ_0 and τ_0 be the standard coordinate functions on $\mathbb{C}^2$, so that $\{d\sigma_0, d\tau_0\}$ is a basis for $H^{1,0}(A_{0,z_0}/\mathbb{C})$ and $\{d\overline{\sigma}_0, d\overline{\tau}_0\}$ is a basis for $H^{0,1}(A_{0,z_0}/\mathbb{C})$. Under the isomorphisms (4.11) these differentials correspond to the smooth 1-forms on $M_2(\mathbb{R})/\mathcal{O}_{B_0}$

$$
(4.12) \qquad
\begin{aligned}
d\sigma_0 &= z_0 da_{11} + da_{12} \\
d\tau_0 &= z_0 da_{21} + da_{22} \\
d\overline{\sigma}_0 &= \overline{z}_0 da_{11} + da_{12} \\
d\overline{\tau}_0 &= \overline{z}_0 da_{21} + da_{22}
\end{aligned}
$$

where a_{ij} are the usual coordinates on $M_2(\mathbb{R})$. The basis of $\mathrm{Lie}(A_{0,z_0}/\mathbb{C})$ dual to the basis $\{d\sigma_0, d\tau_0\}$ of $\mathrm{coLie}(A_{0,z_0}/\mathbb{C})$ is $\{e, f\}$ where

$$
e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
$$

Recall from [12, §3.1] the alternating form ψ_0 on B_0 defined by

$$
\psi_0(x,y) = \frac{1}{\mathrm{disc}(B_0)} \mathrm{Tr}(xsy^*) = \frac{1}{\mathrm{disc}(B_0)} \mathrm{Tr}(xy^t s).
$$

Extend ψ_0 $\mathbb{R}$-linearly to $B_0 \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$, and define a Hermitian form on $M_2(\mathbb{R})$ (with respect to the complex structure determined by J_{z_0})

$$
H_{z_0}(x,y) = \pm\big(\psi_0(xJ_{z_0}, y) + i\psi_0(x,y)\big)
$$

where the sign is chosen so that H_{z_0} is positive definite. As z_0 varies the sign for which this holds is constant on each connected component of X_0, and is different on the two components; by replacing s by $-s$ if necessary, we may assume that the sign is $+1$ on the component with $\mathrm{Im}(z_0) > 0$, and to simplify notation we assume from now on that $\mathrm{Im}(z_0) > 0$. Using (4.11) to identify

$$M_2(\mathbb{R}) \cong \mathbb{C}^2 \cong \mathrm{Lie}(A_{0,z_0}/\mathbb{C}),$$

we view H_{z_0} as a nondegenerate Riemann form on $\mathrm{Lie}(A_{0,z_0}/\mathbb{C})$. If we identify $H^{0,1}(A_{0,z_0}/\mathbb{C})$ with the space of conjugate linear functionals on $\mathrm{Lie}(A_{0,z_0}/\mathbb{C})$ then the $\mathbb{C}$-linear isomorphism

$$\mathrm{Lie}(A_{0,z_0}/\mathbb{C}) \xrightarrow{\lambda_{0,z_0}} \mathrm{Lie}(A_{0,z_0}^\vee/\mathbb{C})$$

factors as

$$\mathrm{Lie}(A_{0,z_0}/\mathbb{C}) \to H^{0,1}(A_{0,z_0}/\mathbb{C}) \to \mathrm{Lie}(A_{0,z_0}^\vee/\mathbb{C})$$

where the first arrow takes the vector v to the conjugate linear functional

$$w \mapsto \pi H_{z_0}(v, w).$$

The factor of $\pi = (2\pi i)/(2i)$ appears because of the "experimental error" of $2i$ at the bottom of [22, p. 87] and the fact that our exponential sequence is shifted from Mumford's by a factor of $2\pi i$. Direct calculation now shows that

$$
\begin{array}{rcl}
\pi H_{z_0}(se, e) = & 0 & = -\pi y_0^{-1} \cdot d\overline{\tau}_0(e) \\
\pi H_{z_0}(se, f) = & -\pi y_0^{-1} & = -\pi y_0^{-1} \cdot d\overline{\tau}_0(f) \\
\pi H_{z_0}(sf, e) = & -\pi y_0^{-1} & = -\pi y_0^{-1} \cdot d\overline{\sigma}_0(e) \\
\pi H_{z_0}(sf, f) = & 0 & = -\pi y_0^{-1} \cdot d\overline{\sigma}_0(e).
\end{array}
$$

This implies that the composition

$$\mathrm{Lie}(A_{0,z_0}/\mathbb{C}) \xrightarrow{s} \mathrm{Lie}(A_{0,z_0}/\mathbb{C}) \xrightarrow{\lambda_{0,z_0}} \mathrm{Lie}(A_{0,z_0}^\vee/\mathbb{C})$$

used in the proof of Proposition 4.2 satisfies

$$e \mapsto \frac{-\pi}{y_0} \cdot d\overline{\tau}_0 \qquad f \mapsto \frac{-\pi}{y_0} \cdot d\overline{\sigma}_0.$$

We next compute the Gauss-Manin connection

$$\nabla(d/dz_0) : H^1_{\mathrm{DR}}(A_{0,z_0}/\mathbb{C}) \to H^1_{\mathrm{DR}}(A_{0,z_0}/\mathbb{C}).$$

Differentiating the equations (4.12) with respect to z_0 shows that $\nabla(d/dz_0)$ satisfies

$$
\begin{array}{rcl}
d\sigma_0 & \mapsto & \dfrac{1}{z_0 - \overline{z}_0}(d\sigma_0 - d\overline{\sigma}_0) \\[2mm]
d\tau_0 & \mapsto & \dfrac{1}{z_0 - \overline{z}_0}(d\tau_0 - d\overline{\tau}_0).
\end{array}
$$

The image

$$\phi_{z_0} \in \mathrm{Hom}_{B_0 \otimes_{\mathbb{Q}} \mathbb{C}}(\mathrm{coLie}(A_{0,z_0}/\mathbb{C}), \mathrm{Lie}(A_{0,z_0}^\vee/\mathbb{C}))$$

of d/dz_0 under the Kodaira-Spencer isomorphism (4.3) is equal to the composition

$$H^{1,0}(A_{0,z_0}/\mathbb{C}) \to H^1_{\mathrm{DR}}(A_{0,z_0}/\mathbb{C}) \xrightarrow{\nabla(d/dz_0)} H^1_{\mathrm{DR}}(A_{0,z_0}/\mathbb{C}) \to H^{0,1}(A_{0,z_0}/\mathbb{C})$$

and so has the explicit form

$$\phi_{z_0}(d\sigma_0) = \frac{-1}{2iy_0} \cdot d\overline{\sigma}_0 \qquad \phi_{z_0}(d\tau_0) = \frac{-1}{2iy_0} \cdot d\overline{\tau}_0.$$

The map Φ_{z_0} making the diagram

$$
\begin{array}{ccc}
\mathrm{coLie}(A_{0,z_0}/\mathbb{C}) & & \\
\Big\downarrow{\scriptstyle \Phi_{z_0}} & \searrow{\scriptstyle \phi_{z_0}} & \\
\mathrm{Lie}(A_{0,z_0}/\mathbb{C}) \xrightarrow{\ s\cdot\ } & \mathrm{Lie}(A_{0,z_0}/\mathbb{C}) \xrightarrow{\ \lambda_{0,z_0}\ } & \mathrm{Lie}(A_{0,z_0}^\vee/\mathbb{C})
\end{array}
$$

commute is then

$$
\Phi_{z_0}(d\sigma_0) = \frac{1}{2\pi i} f \qquad \Phi_{z_0}(d\tau_0) = \frac{1}{2\pi i} e,
$$

and the pairing $Q_{\phi_{z_0}}$ on $\mathrm{coLie}(A_{0,z_0}/\mathbb{C})$ defined in the proof of Proposition 4.2 is completely determined by the single value

$$
Q_{\phi_{z_0}}(d\sigma_0, d\tau_0) = \langle d\sigma_0, \Phi_{z_0}(d\tau_0)\rangle = \frac{1}{2\pi i}\langle d\sigma_0, e\rangle = \frac{1}{2\pi i}.
$$

We deduce that the isomorphism $\omega_0 \cong \omega_0^{\mathrm{Hdg}}$ of Proposition 4.2, when pulled back to an isomorphism of line bundles on X_0, satisfies

$$
dz_0 \mapsto 2\pi i \cdot d\sigma_0 \wedge d\tau_0.
$$

Applying (4.7) shows that

$$
||dz_0||^2 = 4\pi^2 \cdot ||\epsilon_0||^2 = \frac{1}{\pi e^{\gamma^{\mathrm{Euler}}}} \cdot \mathrm{Im}(z_0)^2 \cdot \mathrm{disc}(B_0)
$$

as desired. Similar calculations show that the isomorphism $\omega \cong \omega^{\mathrm{Hdg}}$ of Proposition 4.2, when pulled back to an isomorphism of line bundles on X, satisfies

$$
dz_1 \wedge dz_2 \mapsto \frac{(2\pi i)^2}{\delta_F}(d\sigma_1 \wedge d\tau_1) \wedge (d\sigma_2 \wedge d\tau_2),
$$

and hence (4.9) implies

$$
||dz_1 \wedge dz_2||^2 = \frac{16\pi^4}{d_F} \cdot ||\epsilon||^2 = \frac{d_F}{\pi^2 e^{2\gamma^{\mathrm{Euler}}}} \cdot \mathrm{Im}(z_1)^2 \mathrm{Im}(z_2)^2 \mathrm{disc}(B_0)^2.
$$

$\square$

5. The adjunction formula

In this section we prove the *arithmetic adjunction formula*, Theorem 5.6. This theorem gives an explicit formula for the linear functional (2.8) evaluated at a horizontal irreducible arithmetic cycle on $\mathcal{M}$ intersecting $\mathcal{M}_0$ improperly (*i.e.* completely contained in $\mathcal{M}_0$). This formula is one the main ingredients in the proof of Theorem A.

We return to the notation of §3. Fix a totally positive $v \in F \otimes_{\mathbb{Q}} \mathbb{R}$ and write (v_1, v_2) for the image of v in $\mathbb{R} \times \mathbb{R}$. Similarly, for any $\gamma \in \Gamma$, write (γ_1, γ_2) for the image of γ in $G_0(\mathbb{R}) \times G_0(\mathbb{R})$. For an irreducible horizontal cycle $\mathcal{D}$ of codimension two on $\mathcal{M}$, define a Green current for $\mathcal{D}$

$$
\Xi(\mathcal{D}, v) = \sum_{P \in \mathcal{D}(\mathbb{C})} e_P^{-1} \sum_{\gamma \in \Gamma} \mathbf{g}(\gamma x, v).
$$

On the right $x \in X$ is any point lying above $P \in \mathcal{M}(\mathbb{C})$ under the orbifold presentation (3.2) of $\mathcal{M}(\mathbb{C})$. Denote by

$$
(5.1) \qquad\qquad \widehat{\mathcal{D}}(v) \in \widehat{\mathrm{CH}}^2(\mathcal{M})
$$

the arithmetic cycle class of the pair $(\mathcal{D}, \Xi(\mathcal{D}, v))$. Given a metrized line bundle $\widehat{\mathcal{F}}$ on $\mathcal{M}_0$ and an irreducible cycle $j : \mathcal{D} \to \mathcal{M}_0$ of codimension one, the *Arakelov degree*

$$\widehat{\deg}(\mathcal{D}, j^*\widehat{\mathcal{F}}) \tag{5.2}$$

is defined in [20, Chapter 2], and the *Arakelov height*

$$h_{\widehat{\mathcal{F}}} : Z^1(\mathcal{M}_0) \to \mathbb{R}$$

is defined by linearly extending $\mathcal{D} \mapsto \widehat{\deg}(\mathcal{D}, j^*\widehat{\mathcal{F}})$ to all codimension one cycles with rational coefficients. If instead $\widehat{\mathcal{F}}$ is a metrized line bundle on $\mathcal{M}$ and $j : \mathcal{D} \to \mathcal{M}$ is an irreducible cycle of codimension two, then $\widehat{\deg}(\mathcal{D}, j^*\widehat{\mathcal{F}})$ is defined in the same way as (5.2), and

$$h_{\widehat{\mathcal{F}}} : Z^2(\mathcal{M}) \to \mathbb{R}$$

is the $\mathbb{Q}$-linear extension of $\mathcal{D} \mapsto \widehat{\deg}(\mathcal{D}, j^*\widehat{\mathcal{F}})$.

Lemma 5.1. *Suppose $w \in X$ and $\gamma \in \Gamma$ satisfy both $w \in X_0$ and $\gamma w \in X_0$. Then $\gamma \in \Gamma_0$.*

Proof. Pick any two points $w, w' \in X_0$ and let $P_0 = (\mathbf{A}_{0,w}, \lambda_{0,w})$ and $P_0' = (\mathbf{A}_{0,w'}, \lambda_{0,w'})$ be the objects of $\mathcal{M}_0(\mathbb{C})$ constructed in §3. Thus there is a canonical bijection

$$\{\gamma_0 \in \Gamma_0 : \gamma_0 w = w'\} \cong \mathrm{Iso}_{\mathcal{M}_0(\mathbb{C})}(P_0, P_0'). \tag{5.3}$$

If we then let P and P' be the images of P_0 and P_0' in $\mathcal{M}(\mathbb{C})$, there is a canonical bijection

$$\{\gamma \in \Gamma : \gamma w = w'\} \cong \mathrm{Iso}_{\mathcal{M}(\mathbb{C})}(P, P'). \tag{5.4}$$

According to [12, Lemma 3.1.1] the evident function from the right hand side of (5.3) to the right hand side of (5.4) is a bijection, and hence so is the evident function

$$\{\gamma_0 \in \Gamma_0 : \gamma_0 w = w'\} \to \{\gamma \in \Gamma : \gamma w = w'\}.$$

$\square$

For any $z_0 \in X_0$ define

$$\vartheta_u(z_0) = \sum_{\substack{\gamma \in \Gamma_0 \backslash \Gamma \\ \gamma \notin \Gamma_0}} g_u^0(\gamma_1 z_0, \gamma_2 z_0).$$

The preceding lemma implies that $\gamma_1 z_0 \neq \gamma_2 z_0$ in each term on the right, so that each term in the infinite sum is defined. Using the methods of [9, §6.5–6.6] and the rapid decay of g_u^0 away from the diagonal [20, Remark 7.3.2] one can show that the summation converges uniformly on compact subsets of X_0, and so defines a smooth function on the orbifold $[\Gamma_0 \backslash X_0]$. For a positive $u \in \mathbb{R}$, denote by $\widehat{\mathcal{O}}_{\mathcal{M}_0}(u)$ the structure sheaf of $\mathcal{M}_0$, endowed with the metric defined by

$$-\log \|1\|_P^2 = \vartheta_u(P)$$

for every $P \in \mathcal{M}_0(\mathbb{C})$. Here 1 denotes the constant function 1 on the orbifold $\mathcal{M}_0(\mathbb{C})$. For any irreducible cycle $\mathcal{D}_0$ on $\mathcal{M}_0$ we have, from the definition of

Arakelov height, the relation

$$(5.5) \qquad h_{\widehat{\mathcal{O}}_{\mathcal{M}_0}(u)}(\mathcal{D}_0) = \frac{1}{2} \sum_{P \in \mathcal{D}_0(\mathbb{C})} e_P^{-1} \vartheta_u(P).$$

We next metrize the line bundle

$$\mathcal{L} \overset{\text{def}}{=} \mathcal{O}_{\mathcal{M}}(\mathcal{M}_0)$$

on $\mathcal{M}$. If we denote by s the constant function 1 on $\mathcal{M}$ viewed as a global section of $\mathcal{L}$, there is a unique smooth metric $||\cdot||$ on $\mathcal{L}$ satisfying

$$(5.6) \qquad -\log||s||_P^2 = \log(4u \cdot d_F \mathrm{disc}(B_0)) + \sum_{\gamma \in \Gamma_0 \backslash \Gamma} g_u^0(\gamma_1 x_1, \gamma_2 x_2)$$

for every $P \in \mathcal{M}(\mathbb{C}) \smallsetminus \mathcal{M}_0(\mathbb{C})$ and $x \in X$ lying above P. To see that the metric extends smoothly across $\mathcal{M}_0(\mathbb{C})$ one uses [20, (7.3.16)] to show that near a point of X_0 the right hand side has the form

$$g_u^0(x_1, x_2) + \text{smooth} = -\log|x_1 - x_2|^2 + \text{smooth}.$$

Let $\widehat{\mathcal{L}}(u)$ denote the line bundle $\mathcal{L}$ on $\mathcal{M}$ endowed with the above metric. The pullback $i^*\mathcal{L}$ is the *normal bundle* of the closed immersion $i : \mathcal{M}_0 \to \mathcal{M}$ and the classical adjunction formula [21, Theorem 6.4.9] provides a canonical isomorphism

$$(5.7) \qquad \omega_0 \cong i^*\mathcal{L} \otimes i^*\omega$$

in which ω_0 and ω are the canonical bundles on $\mathcal{M}_0$ and $\mathcal{M}$ as in §4. The following proposition is our first form of the arithmetic adjunction formula.

Proposition 5.2. *There is an isomorphism of metrized line bundles on $\mathcal{M}_0$*

$$(5.8) \qquad \widehat{\omega}_0 \otimes \widehat{\mathcal{O}}_{\mathcal{M}_0}(u) \cong i^*\widehat{\mathcal{L}}(u) \otimes i^*\widehat{\omega}.$$

Proof. Let L_0 be the pullback of $i^*\mathcal{L}$ to a line bundle on X_0, so that L_0 is isomorphic to the pullback of $\mathcal{O}_X(X_0)$ to X_0. The function $f(z_1, z_2) = (z_1 - z_2)^{-1}$ on X defines a global nonvanishing section of $\mathcal{O}_X(X_0)$, which in turn restricts to a global nonvanishing section σ_0 of L_0. Under the metric on L_0 determined by the metric (5.6) on $i^*\mathcal{L}$, this section has norm (using [20, (7.3.16)] for the final equality)

$$
\begin{aligned}
-\log||\sigma_0||_{z_0}^2 &= -\lim_{x \to z_0} \log||s/(x_1 - x_2)||_x^2 \\
&= \log(4u \cdot d_F \mathrm{disc}(B_0)) + \vartheta_u(z_0) + \lim_{x \to z_0}(g_u^0(x_1, x_2) + \log|x_1 - x_2|^2) \\
(5.9) \qquad &= \log(d_F \mathrm{disc}(B_0)) + \vartheta_u(z_0) - \gamma_{\mathrm{Euler}} - \log\left(\frac{\pi}{\mathrm{Im}(z_0)^2}\right)
\end{aligned}
$$

where $z_0 \in X_0$, and in each limit $x \in X \smallsetminus X_0$. The isomorphism of line bundles (5.7) can be viewed as an isomorphism

$$(5.10) \qquad \omega_0 \otimes \mathcal{O}_{\mathcal{M}_0} \cong i^*\mathcal{L} \otimes \omega,$$

which pulls back to the isomorphism of line bundles

$$\Omega_{X_0}^1 \otimes \mathcal{O}_{X_0} \cong L_0 \otimes i^*\Omega_X^2$$

on X_0 determined by $dz_0 \otimes 1 \leftrightarrow \sigma_0 \otimes (dz_1 \wedge dz_2)$. Comparing (5.9) with Proposition 4.4 gives

$$
\begin{aligned}
-\log ||dz_0 \otimes 1||_{z_0}^2 &= \gamma_{\mathrm{Euler}} + \log \left(\frac{\pi}{\mathrm{disc}(B_0) \cdot \mathrm{Im}(z_0)^2} \right) + \vartheta_u(z_0) \\
&= -\log ||\sigma_0 \otimes (dz_1 \wedge dz_2)||_{z_0}^2,
\end{aligned}
$$

and therefore the isomorphism (5.10) respects the metrics of (5.8). $\qquad \square$

Corollary 5.3. *There is an isomorphism of metrized line bundles on $\mathcal{M}_0$*

$$
\widehat{\mathcal{O}}_{\mathcal{M}_0}(u) \cong i^* \widehat{\mathcal{L}}(u) \otimes \widehat{\omega}_0.
$$

In particular for any irreducible horizontal cycle $\mathcal{D}$ on $\mathcal{M}_0$

$$
h_{i^* \widehat{\mathcal{L}}(u)}(\mathcal{D}) + h_{\widehat{\omega}_0}(\mathcal{D}) = \frac{1}{2} \sum_{P \in \mathcal{D}(\mathbb{C})} e_P^{-1} \vartheta_u(x_0).
$$

Proof. The first claim is immediate from Proposition 5.2 and the isomorphism of metrized line bundles (4.10). The second claim is then just a restatement of (5.5). $\qquad \square$

Lemma 5.4. *For any irreducible horizontal cycle $\mathcal{D}$ on $\mathcal{M}$ of codimension two*

$$
\begin{aligned}
\widehat{\deg}_{\mathcal{M}_0} \widehat{\mathcal{D}}(v) &= h_{\widehat{\mathcal{L}}(v_1)}(\mathcal{D}) - \frac{1}{2} \deg_{\mathbb{Q}}(\mathcal{D}) \log(4 v_1 d_F \mathrm{disc}(B_0)) \\
&\quad + \frac{1}{2} \sum_{P \in \mathcal{D}(\mathbb{C})} e_P^{-1} \sum_{\gamma \in \Gamma_0 \backslash \Gamma} \int_{X_0} \mathbf{g}_0(\gamma_2 x_2, v_2) \wedge \Phi_0(\gamma_1 x_1, v_1)
\end{aligned}
$$

where $x \in X$ is any point above P under $\mathcal{M}(\mathbb{C}) \cong [\Gamma \backslash X]$, and

(5.11) $$ \deg_{\mathbb{Q}}(\mathcal{D}) = \sum_{P \in \mathcal{D}(\mathbb{C})} e_P^{-1}. $$

Proof. Kudla's function g_u^0 on X satisfies the Green equation

$$
dd^c g_u^0 + \delta_{X_0} = c_u^0
$$

for some smooth Γ_0-invariant $(1,1)$-form c_u^0 on X, and from the explicit calculation of $dd^c g_u^0$ in the proof of [20, Proposition 7.3.1] we see that

$$
c_u^0 = \phi_u^0 \cdot \pi_1^* \mu_0 + \phi_u^0 \cdot \pi_2^* \mu_0 + \alpha_u \cdot dz_1 \wedge d\overline{z}_2 + \beta_u \cdot d\overline{z}_1 \wedge dz_2
$$

for smooth functions α_u and β_u on X. Define a Γ_0-invariant function

$$
G_u^0(x) = \log(4u \cdot d_F \mathrm{disc}(B_0)) + \sum_{\gamma \in \Gamma_0 \backslash \Gamma} g_u^0(\gamma_1 x_1, \gamma_2 x_2)
$$

on $X \smallsetminus \Gamma X_0$, and view G_u^0 as a $(0,0)$-current on the orbifold $\mathcal{M}(\mathbb{C}) \cong [\Gamma \backslash X]$. As G_u^0 satisfies the Green equation

$$
dd^c G_u^0 + \delta_{\mathcal{M}_0} = \sum_{\gamma \in \Gamma_0 \backslash \Gamma} \gamma^* c_u^0,
$$

we may consider the arithmetic cycle class $\widehat{\mathcal{M}}_0(u) \in \widehat{\mathrm{CH}}^1(\mathcal{M})$ determined by $(\mathcal{M}_0, G_u^0)$. Comparing with (5.6), we note that $\widehat{\mathcal{M}}_0(u)$ is the arithmetic Chern class (in the sense of [1, §2.1.2]) of the metrized line bundle $\widehat{\mathcal{L}}(u)$.

Recall from [7, Lemma 3.4.3] that there is a canonical isomorphism of $\mathbb{Q}$-vector spaces

$$\widehat{\mathrm{CH}}^1(\mathrm{Spec}(\mathbb{Z})) \to \mathbb{R}.$$

The lemma is really a special case of [1, Proposition 2.3.1], which relates both sides of the stated equality to the value of $\widehat{\mathcal{D}}(v) \cdot \widehat{\mathcal{M}}_0(v_1)$, where the product is the arithmetic intersection

$$(5.12) \qquad \widehat{\mathrm{CH}}^2(\mathcal{M}) \times \widehat{\mathrm{CH}}^1(\mathcal{M}) \to \widehat{\mathrm{CH}}^1(\mathrm{Spec}(\mathbb{Z})) \to \mathbb{R}.$$

Indeed, what we call the arithmetic degree $\widehat{\deg}_{\mathcal{M}_0}\widehat{D}(v)$ is equal to the intersection pairing $(\widehat{\mathcal{D}}(v) \mid \mathcal{M}_0)$, where the pairing

$$\widehat{\mathrm{CH}}^2(\mathcal{M}) \times Z_2(\mathcal{M}) \to \widehat{\mathrm{CH}}^1(\mathrm{Spec}(\mathbb{Z})) \to \mathbb{R}$$

is defined by [1, (2.3.1)]. On the other hand, by [1, Proposition 2.3.1(vi)] the Arakelov height $h_{\widehat{\mathcal{L}}(u)}(\mathcal{D})$ is equal to the intersection pairing $(\widehat{\mathcal{M}}_0(u) \mid \mathcal{D})$, where now we use the pairing

$$\widehat{\mathrm{CH}}^1(\mathcal{M}) \times Z_1(\mathcal{M}) \to \widehat{\mathrm{CH}}^1(\mathrm{Spec}(\mathbb{Z})) \to \mathbb{R}.$$

Comparing [1, (2.3.1)] with [1, (2.3.3)] shows that

$$\widehat{\deg}_{\mathcal{M}_0}\widehat{\mathcal{D}}(v) = \widehat{\mathcal{D}}(v) \cdot \widehat{\mathcal{M}}_0(u) - \frac{1}{2} \sum_{P \in \mathcal{D}(\mathbb{C})} e_P^{-1} \int_X G_u^0 \wedge \Phi_1(x,v) \wedge \Phi_2(x,v).$$

Taking $u = v_1$ and using

$$\int_X \log(4u\mathrm{disc}(B_0)) \wedge \Phi_1(x,v) \wedge \Phi_2(x,v)$$

$$= \log(4u\mathrm{disc}(B_0)) \left(\int_{X_0} \phi_{v_1}^0(x_1,z_1)d\mu_0(z_1) \right) \left(\int_{X_0} \phi_{v_2}^0(x_2,z_2)d\mu_0(z_2) \right)$$

$$= \log(4u\mathrm{disc}(B_0))$$

shows that $\widehat{\mathcal{D}}(v) \cdot \widehat{\mathcal{M}}_0(v_1)$ is equal to

$$\widehat{\deg}_{\mathcal{M}_0}\widehat{\mathcal{D}}(v) + \frac{1}{2}\deg_{\mathbb{Q}}(\mathcal{D})\log(4v_1\mathrm{disc}(B_0))$$

$$+ \frac{1}{2}\sum_{P \in \mathcal{D}(\mathbb{C})} e_P^{-1} \sum_{\gamma \in \Gamma_0 \backslash \Gamma} \int_X g_{v_1}^0(\gamma_1 z_1, \gamma_2 z_2)\phi_{v_1}^0(x_1,z_1)\phi_{v_2}^0(x_2,z_2)d\mu_0(z_1)d\mu_0(z_2).$$

On the other hand, we may use the symmetry of the pairing (5.12) to reverse the roles of $\mathcal{M}_0$ and $\mathcal{D}$, and deduce that

$$h_{\widehat{\mathcal{L}}(v_1)}(\mathcal{D}) = \widehat{\mathcal{M}}_0(v_1) \cdot \widehat{\mathcal{D}}(v) - \frac{1}{2}\sum_{P \in \mathcal{D}(\mathbb{C})} e_P^{-1} \sum_{\gamma \in \Gamma_0 \backslash \Gamma} \int_X \gamma^* c_{v_1}^0 \wedge \mathbf{g}(x,v).$$

The integral can be rewritten, using $(\gamma^{-1})^* \mathbf{g}(x, v) = \mathbf{g}(\gamma x, v)$, as

$$\int_X c_{v_1}^0 \wedge \mathbf{g}(\gamma x, v)$$

$$= \int_{\{\gamma_1 x_1\} \times X_0} c_{v_1}^0 \wedge \mathbf{g}_2(\gamma x, v) + \int_X c_{v_1}^0 \wedge \mathbf{g}_1(\gamma x, v) \wedge \Phi_2(\gamma x, v)$$

$$= \int_{\{\gamma_1 x_1\} \times X_0} \phi_{v_1}^0 \pi_2^* \mu_0 \wedge \mathbf{g}_2(\gamma x, v) + \int_X \phi_{v_1}^0 \pi_1^* \mu_0 \wedge \mathbf{g}_1(\gamma x, v) \wedge \Phi_2(\gamma x, v)$$

$$= \int_{X_0} \phi_{v_1}^0(\gamma_1 x_1, z_2) g_{v_2}^0(\gamma_2 x_2, z_2) d\mu_0(z_2)$$

$$+ \int_X g_{v_1}^0(\gamma_1 x_1, z_1) \phi_{v_1}^0(z_1, z_2) \phi_{v_2}^0(\gamma_2 x_2, z_2) d\mu_0(z_1) d\mu_0(z_2)$$

$$= \int_{X_0} \mathbf{g}_0(\gamma_2 x_2, v_2) \wedge \Phi_0(\gamma_1 x_1, v_1)$$

$$+ \int_X g_{v_1}^0(z_1, z_2) \phi_{v_1}^0(z_1, \gamma_1 x_2) \phi_{v_2}^0(\gamma_2 x_2, z_2) d\mu_0(z_1) d\mu_0(z_2)$$

$$= \int_{X_0} \mathbf{g}_0(\gamma_2 x_2, v_2) \wedge \Phi_0(\gamma_1 x_1, v_1)$$

$$+ \int_X g_{v_1}^0(\gamma_1 z_1, \gamma_2 z_2) \phi_{v_1}^0(z_1, x_2) \phi_{v_2}^0(x_2, z_2) d\mu_0(z_1) d\mu_0(z_2).$$

For the second to last equality we have used the following observation: for each fixed $z_2 \in X_0$ there is a $T \in G_0(\mathbb{R})$ whose action on X_0 interchanges z_2 and $\gamma_1 x_1$. Hence

$$\int_{X_0} g_{v_1}(\gamma_1 x_1, z_1) \phi_{v_1}(z_1, z_2) d\mu_0(z_1) = \int_{X_0} g_{v_1}(z_2, T^{-1} z_1) \phi_{v_1}(z_1, z_2) d\mu_0(z_1)$$

$$= \int_{X_0} g_{v_1}(z_2, z_1) \phi_{v_1}(T z_1, z_2) d\mu_0(z_1)$$

$$= \int_{X_0} g_{v_1}(z_2, z_1) \phi_{v_1}(z_1, \gamma_1 x_1) d\mu_0(z_1).$$

Comparing the two formulas for $\widehat{\mathcal{D}}(v) \cdot \widehat{\mathcal{M}}_0(v_1) = \widehat{\mathcal{M}}_0(v_1) \cdot \widehat{\mathcal{D}}(v)$ proves the claim. $\square$

Remark 5.5. The proof of Lemma 5.4 makes extensive use of the arithmetic intersection theory for *schemes* developed in [7] and in [1], while we are working with the *stacks* $\mathcal{M}_0$ and $\mathcal{M}$. This can be justified by using a trick of Bruinier-Burgos-Kühn [3] to deduce an adequate intersection theory for $\mathcal{M}$ by writing (for every $N \in \mathbb{Z}^+$) the stack $\mathcal{M}_{/\mathbb{Z}[1/N]}$ as the quotient of a $\mathbb{Z}[1/N]$-scheme $M_{\mathbb{Z}[1/N]}$ by the action of a finite group, and using compatibility of the Gillet-Soulé intersection theory for $M_{\mathbb{Z}[1/N]}$ as N varies. See for example the construction of $\widehat{\deg}_{\mathcal{M}_0}$ given in [12, §2.3].

Now suppose we start with an irreducible horizontal cycle $\mathcal{D}$ of codimension one on $\mathcal{M}_0$. Viewing $\mathcal{D}$ as a codimension two cycle on $\mathcal{M}$, we may form the arithmetic cycle class $\widehat{\mathcal{D}}(v)$ of (5.1). Our final form of the arithmetic adjunction formula will compute the arithmetic degree of $\widehat{\mathcal{D}}(v)$ along $\mathcal{M}_0$ in terms of purely archimedean data and the quantity $h_{\widehat{\omega}_0}(\mathcal{D})$. The essential point is that while the arithmetic degree of $\widehat{\mathcal{D}}(v)$ along $\mathcal{M}_0$ depends on the positions of $\mathcal{M}_0$ and $\mathcal{D}$ inside of the ambient threefold $\mathcal{M}$, the Arakelov height $h_{\widehat{\omega}_0}(\mathcal{D})$ depends only on the position of

$\mathcal{D}$ in $\mathcal{M}_0$, and makes no reference to the threefold $\mathcal{M}$. In the applications $\mathcal{D}$ will be chosen in such a way that the quantity $h_{\widehat{\omega}_0}(\mathcal{D})$ has already been calculated by Kudla-Rapoport-Yang [19]. To state the formula we need the following notation. As in [20, Lemma 7.5.4] define a function $J : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$J(t) = \int_0^\infty w^{-1} e^{-tw} [(w+1)^{1/2} - 1] \, dw$$

so that for any fixed $x_0 \in X_0$

$$(5.13) \qquad \int_{X_0} \mathbf{g}_0(x_0, v_2) \wedge \Phi_0(x_0, v_1) = \log\left(\frac{v_1 + v_2}{v_2}\right) - J(4\pi v_1 + 4\pi v_2).$$

Theorem 5.6 (Arithmetic adjunction). *Suppose $\mathcal{D}$ is an irreducible horizontal cycle of codimension one on $\mathcal{M}_0$. Viewing $\mathcal{D}$ as a cycle on $\mathcal{M}$, let*

$$\widehat{\mathcal{D}}(v) \in \widehat{\mathrm{CH}}^2(\mathcal{M})$$

be the arithmetic cycle class of (5.1). Then

$$
\begin{aligned}
h_{\widehat{\omega}_0}(\mathcal{D}) + \widehat{\deg}_{\mathcal{M}_0} \widehat{\mathcal{D}}(v) \;=\; & \frac{1}{2} \deg_{\mathbb{Q}}(\mathcal{D}) \cdot \log\left(\frac{v_1 + v_2}{4 v_1 v_2 \cdot d_F \mathrm{disc}(B_0)}\right) \\
& - \frac{1}{2} \deg_{\mathbb{Q}}(\mathcal{D}) \cdot J(4\pi v_1 + 4\pi v_2) \\
& + \frac{1}{2} \sum_{P \in \mathcal{D}(\mathbb{C})} e_P^{-1} \sum_{\substack{\gamma \in \Gamma_0 \backslash \Gamma \\ \gamma \notin \Gamma_0}} \int_{X_0} \mathbf{g}_0(\gamma_1 x_0, v_1) * \mathbf{g}_0(\gamma_2 x_0, v_2).
\end{aligned}
$$

In the integral $x_0 \in X_0$ is any point above P under $\mathcal{M}_0(\mathbb{C}) \cong [\Gamma_0 \backslash X_0]$, and $\deg_{\mathbb{Q}}$ is defined by (5.11).

Proof. The crux of the proof is a trivial observation: the height $h_{\widehat{\mathcal{L}}(u)}(\mathcal{D})$ depends only on the pullback of the metrized line bundle $\widehat{\mathcal{L}}(u)$ to $\mathcal{D}$, which can be computed by first pulling back from $\mathcal{M}$ to $\mathcal{M}_0$ and then from $\mathcal{M}_0$ to $\mathcal{D}$. In other words

$$h_{i^* \widehat{\mathcal{L}}(u)}(\mathcal{D}) = h_{\widehat{\mathcal{L}}(u)}(\mathcal{D})$$

where $\mathcal{D}$ is regarded as a cycle on $\mathcal{M}_0$ on the left, and as a cycle on $\mathcal{M}$ on the right. Thus Lemma 5.4 shows that $\widehat{\deg}_{\mathcal{M}_0} \widehat{\mathcal{D}}(v)$ is equal to

$$
\begin{aligned}
& h_{i^* \widehat{\mathcal{L}}(v_1)}(\mathcal{D}) - \frac{1}{2} \deg_{\mathbb{Q}}(\mathcal{D}) \log(4 v_1 d_F \mathrm{disc}(B_0)) \\
& \quad + \frac{1}{2} \sum_{P \in \mathcal{D}(\mathbb{C})} e_P^{-1} \sum_{\gamma \in \Gamma_0 \backslash \Gamma} \int_{X_0} \mathbf{g}_0(\gamma_2 x_0, v_2) \wedge \Phi_0(\gamma_1 x_0, v_1).
\end{aligned}
$$

Using Corollary 5.3 and (5.13) this can be written as

$$
\begin{aligned}
& -h_{\widehat{\omega}_0}(\mathcal{D}) + \frac{1}{2} \deg_{\mathbb{Q}}(\mathcal{D}) \log\left(\frac{v_1 + v_2}{4 v_1 v_2 \cdot d_F \mathrm{disc}(B_0)}\right) - \frac{1}{2} \deg_{\mathbb{Q}}(\mathcal{D}) J(4\pi v_1 + 4\pi v_2) \\
& \quad + \frac{1}{2} \sum_{P \in \mathcal{D}(\mathbb{C})} e_P^{-1} \left(\vartheta_{v_1}(x_0) + \sum_{\substack{\gamma \in \Gamma_0 \backslash \Gamma \\ \gamma \notin \Gamma_0}} \int_{X_0} \mathbf{g}_0(\gamma_2 x_0, v_2) \wedge \Phi_0(\gamma_1 x_0, v_1) \right).
\end{aligned}
$$

The final quantity in parentheses is

$$\sum_{\substack{\gamma \in \Gamma_0 \backslash \Gamma \\ \gamma \notin \Gamma_0}} \left(g^0_{v_1}(\gamma_1 x_0, \gamma_2 x_0) + \int_{X_0} \mathbf{g}_0(\gamma_2 x_0, v_2) \wedge \Phi_0(\gamma_1 x_0, v_1) \right)$$

$$= \sum_{\substack{\gamma \in \Gamma_0 \backslash \Gamma \\ \gamma \notin \Gamma_0}} \int_{X_0} \left(\mathbf{g}_0(\gamma_1 x_0, v_1) \wedge \delta_{\{\gamma_2 x_0\}} + \mathbf{g}_0(\gamma_2 x_0, v_2) \wedge \Phi_0(\gamma_1 x_0, v_1) \right)$$

$$= \sum_{\substack{\gamma \in \Gamma_0 \backslash \Gamma \\ \gamma \notin \Gamma_0}} \int_{X_0} \mathbf{g}_0(\gamma_1 x_0, v_1) * \mathbf{g}_0(\gamma_2 x_0, v_2)$$

completing the proof. $\square$

6. Unramified intersection theory

The arithmetic adjunction formula of the previous section will allow us to compute the degree along $\mathcal{M}_0$ of those horizontal components of $\mathcal{Y}(\alpha)$ that intersect $\mathcal{M}_0$ improperly, and now we turn to the intersection theory of the remaining components of $\mathcal{Y}(\alpha)$. Much of the hard work in these calculations is contained in [11]. In this section we consider intersection multiplicities in characteristics prime to the discriminant of B_0. The case of characteristic dividing the discriminant of B_0 will be treated in the next section.

Fix a totally positive $\alpha \in \mathcal{O}_F$ and abbreviate $\mathcal{Y} = \mathcal{Y}(\alpha)$ and $\mathcal{Y}_0 = \mathcal{Y}_0(\alpha)$. Let p be a prime that does not divide the discriminant of B_0, and fix an isomorphisms of stacks $\mathcal{M}_{/\mathbb{Z}_p} \cong [H \backslash M]$ with M a $\mathbb{Z}_p$-scheme and H a finite group of automorphisms of M. Set

$$Y = \mathcal{Y} \times_{\mathcal{M}} M \qquad M_0 = \mathcal{M}_0 \times_{\mathcal{M}} M \qquad Y_0 = \mathcal{Y}_0 \times_{\mathcal{M}} M$$

so that there is a cartesian diagram of $\mathbb{Z}_p$-schemes

$$\begin{array}{ccc} Y_0 & \xrightarrow{j} & Y \\ {\scriptstyle \phi_0} \downarrow & & \downarrow {\scriptstyle \phi} \\ M_0 & \xrightarrow{i} & M. \end{array}$$

The scheme Y has dimension at most one (see [12, Proposition 3.3.1]). The scheme Y_0 has dimension zero if $F(\sqrt{-\alpha})/\mathbb{Q}$ is not biquadratic (see the proof of [12, Lemma 5.1.2]) but otherwise Y_0 may have components of dimension one. For every nonzero $T \in \mathrm{Sym}_2(\mathbb{Z})^\vee$ and nonzero $t \in \mathbb{Z}$ set

$$Z(T) = \mathcal{Z}(T) \times_{\mathcal{M}_0} M_0 \qquad Z(t) = \mathcal{Z}(t) \times_{\mathcal{M}_0} M_0.$$

If $\det(T) \neq 0$ then the scheme $Z(T)$ is zero dimensional (by [20, Theorem 3.6.1]). If $\det(T) = 0$ then $Z(T)$ has dimension at most one and every irreducible component is horizontal (by [20, Proposition 3.4.5] and [20, Lemma 6.4.1]). The scheme $Z(t)$ has dimension at most one and every irreducible component is horizontal (again by [20, Proposition 3.4.5]). The decomposition (2.2) induces a decomposition

(6.1) $$Y_0 = \bigsqcup_{T \in \Sigma(\alpha)} Z(T).$$

For any Noetherian scheme X let $\mathbf{K}_0(X)$ be the Grothendieck group of the category of coherent $\mathcal{O}_X$-modules. If $J \to X$ is a proper morphism, let $\mathbf{K}_0^J(X)$ be the Grothendieck group of the category of coherent $\mathcal{O}_X$- modules that are supported on the image of J. If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module we denote by $[\mathcal{F}]$ the corresponding class in $\mathbf{K}_0(X)$. Let $\Pi(Y)$ denote the set of all irreducible components of Y of dimension one, and endow each such component with its reduced subscheme structure. If $D \in \Pi(Y)$ has generic point η, and $\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module, define the *multiplicity of $\mathcal{F}$ along D* to be the length of the stalk

$$\mathrm{mult}_D(\mathcal{F}) = \mathrm{length}_{\mathcal{O}_{Y,\eta}}(\mathcal{F}_\eta).$$

The multiplicity is finite (as $\mathcal{O}_{Y,\eta}$ is Artinian) and depends only on the class of $\mathcal{F}$ in $\mathbf{K}_0(Y)$, not on the sheaf $\mathcal{F}$ itself. Define

$$[\mathcal{F}]_D = \mathrm{mult}_D(\mathcal{F}) \cdot [\mathcal{O}_D] \in \mathbf{K}_0(D).$$

As the inclusion $D \to Y$ is finite, push forward of sheaves is an exact functor from coherent $\mathcal{O}_D$-modules to coherent $\mathcal{O}_Y$-modules, and induces a homomorphism $\mathbf{K}_0(D) \to \mathbf{K}_0(Y)$. Thus we may also view $[\mathcal{F}]_D \in \mathbf{K}_0(Y)$. It follows from [12, Lemma 2.2.2] that for any class $[\mathcal{F}] \in \mathbf{K}_0(Y)$ there is a canonical decomposition in $\mathbf{K}_0(Y)$

$$(6.2) \qquad [\mathcal{F}] = [\mathcal{F}]^{\mathrm{small}} + \sum_{D \in \Pi(Y)} [\mathcal{F}]_D$$

in which $[\mathcal{F}]^{\mathrm{small}}$ lies in the image of $\mathbf{K}_0^J(Y) \to \mathbf{K}_0(Y)$ for some closed subscheme $J \to Y$ of dimension zero.

Definition 6.1. We say that an irreducible component $D \in \Pi(Y)$ is *improper* if it is contained in the closed subscheme Y_0, and is *proper* otherwise. Thus the term "proper" is shorthand for "meets M_0 properly."

Let $\Pi^{\mathrm{prop}}(Y) \subset \Pi(Y)$ be the subset of proper components, and write $\Pi(Y)$ as a disjoint union

$$\Pi(Y) = \Pi^\bullet(Y) \cup \Pi^{\bullet\bullet}(Y) \cup \Pi^{\mathrm{ver}}(Y)$$

in which $\Pi^{\mathrm{ver}}(Y)$ is the subset of vertical components, $\Pi^\bullet(Y)$ is the subset of proper horizontal components, and $\Pi^{\bullet\bullet}(Y)$ is the subset of improper horizontal components. As Y_0 has no vertical components of dimension one (by the decomposition (6.1) and the corresponding property of $Z(T)$ noted above)

$$\Pi^{\mathrm{prop}}(Y) = \Pi^\bullet(Y) \cup \Pi^{\mathrm{ver}}(Y).$$

Give

$$Y^\bullet = \bigcup_{D \in \Pi^\bullet(Y)} D \subset Y$$

its reduced subscheme structure and define $Y^{\bullet\bullet}$, Y^{ver}, and Y^{prop} similarly. For any $[\mathcal{F}] \in \mathbf{K}_0(Y)$, set

$$[\mathcal{F}]^\bullet = \sum_{D \in \Pi^\bullet(Y)} [\mathcal{F}]_D \in \mathbf{K}_0(Y^\bullet)$$

and define $[\mathcal{F}]^{\bullet\bullet}$, $[\mathcal{F}]^{\mathrm{ver}}$, and $[\mathcal{F}]^{\mathrm{prop}}$ similarly. In $\mathbf{K}_0(Y)$ we have the relation

$$[\mathcal{F}] = [\mathcal{F}]^{\mathrm{small}} + [\mathcal{F}]^\bullet + [\mathcal{F}]^{\bullet\bullet} + [\mathcal{F}]^{\mathrm{ver}},$$

and in $\mathbf{K}_0(Y^{\mathrm{prop}})$ we have

$$(6.3) \qquad [\mathcal{F}]^{\mathrm{prop}} = [\mathcal{F}]^\bullet + [\mathcal{F}]^{\mathrm{ver}}.$$

For any coherent $\mathcal{O}_{Y^{\mathrm{prop}}}$-module $\mathcal{F}$ (which we also view as a coherent $\mathcal{O}_Y$-module supported on Y^{prop}) and any $y \in Y_0(\mathbb{F}_p^{\mathrm{alg}})$ (which we also view as an element of $Y(\mathbb{F}_p^{\mathrm{alg}})$, $M_0(\mathbb{F}_p^{\mathrm{alg}})$, or $M(\mathbb{F}_p^{\mathrm{alg}})$ as needed) define

$$(6.4) \qquad \begin{aligned} I_{\mathcal{O}_{Y_0,y}}(\mathcal{F}, \mathcal{O}_{M_0}) &= \sum_{\ell \geq 0}(-1)^\ell \cdot \mathrm{length}_{\mathcal{O}_{Y_0,y}} \mathrm{Tor}_\ell^{\mathcal{O}_{M,y}}(\mathcal{F}_y, \mathcal{O}_{M_0,y}) \\ &= \sum_{\ell \geq 0}(-1)^\ell \cdot \mathrm{length}_{\mathcal{O}_{Y_0,y}} \mathrm{Tor}_\ell^{\mathcal{O}_{Y,y}}(\mathcal{F}_y, \mathcal{O}_{Y_0,y}). \end{aligned}$$

The rule $[\mathcal{F}] \mapsto I_{\mathcal{O}_{Y_0,y}}(\mathcal{F}, \mathcal{O}_{M_0})$ determines a homomorphism $\mathbf{K}_0(Y^{\mathrm{prop}}) \to \mathbb{Z}$.

Proposition 6.2. *For every nonsingular $T \in \Sigma(\alpha)$ and $y \in Z(T)(\mathbb{F}_p^{\mathrm{alg}})$*

$$I_{\mathcal{O}_{Y_0,y}}([\mathcal{O}_Y]^\bullet, \mathcal{O}_{M_0}) + I_{\mathcal{O}_{Y_0,y}}([\mathcal{O}_Y]^{\mathrm{ver}}, \mathcal{O}_{M_0}) = \mathrm{length}_{\mathcal{O}_{Z(T),y}}(\mathcal{O}_{Z(T),y}).$$

Proof. For any $y \in Z(T)(\mathbb{F}_p^{\mathrm{alg}})$, the decomposition (6.1) implies $\mathcal{O}_{Y_0,y} \cong \mathcal{O}_{Z(T),y}$. As noted earlier, the hypothesis that T is nonsingular implies that $Z(T)$ has dimension zero, and hence $\mathcal{O}_{Y_0,y}$ is Artinian. It follows that the right hand side of (6.4) is defined for *every* coherent $\mathcal{O}_Y$-module $\mathcal{F}$, not merely for coherent $\mathcal{O}_{Y^{\mathrm{prop}}}$-modules, and that

$$\mathcal{F} \mapsto I_{\mathcal{O}_{Y_0,y}}(\mathcal{F}, \mathcal{O}_{M_0})$$

extends to a homomorphism $\mathbf{K}_0(Y) \to \mathbb{Z}$. Furthermore [12, Lemma 4.2.2] implies that $[\mathcal{F}]^{\mathrm{small}}$ lies in the kernel of this homomorphism, and so (6.2) implies

$$(6.5) \qquad I_{\mathcal{O}_{Y_0,y}}(\mathcal{F}, \mathcal{O}_{M_0}) = \sum_{D \in \Pi(Y)} I_{\mathcal{O}_{Y_0,y}}([\mathcal{F}]_D, \mathcal{O}_{M_0}).$$

If $D \in \Pi(Y)$ is an improper component then $D \subset Y_0$, and so

$$\dim Z(T) = 0 \implies D \not\subset Z(T) \implies y \notin D(\mathbb{F}_p^{\mathrm{alg}}).$$

Hence if we view $\mathcal{O}_D$ as a coherent $\mathcal{O}_M$-module, the stalk $\mathcal{O}_{D,y}$ is trivial. Taking $\mathcal{F} = \mathcal{O}_Y$ in (6.5) now gives

$$(6.6) \qquad I_{\mathcal{O}_{Y_0,y}}(\mathcal{O}_Y, \mathcal{O}_{M_0}) = I_{\mathcal{O}_{Y_0,y}}([\mathcal{O}_Y]^{\mathrm{prop}}, \mathcal{O}_{M_0}).$$

The local ring $\mathcal{O}_{Y,y}$ is Cohen-Macaulay of dimension one by [12, Lemma 3.3.4] and [12, Corollary 3.3.9], and so by the argument leading to (2.5)

$$\mathrm{Tor}_\ell^{\mathcal{O}_{M,y}}(\mathcal{O}_{Y,y}, \mathcal{O}_{M_0,y}) = 0$$

for $\ell > 0$. Therefore

$$I_{\mathcal{O}_{Y_0,y}}(\mathcal{O}_Y, \mathcal{O}_{M_0}) = \mathrm{length}_{\mathcal{O}_{Y_0,y}}(\mathcal{O}_{Y_0,y}) = \mathrm{length}_{\mathcal{O}_{Z(T),y}}(\mathcal{O}_{Z(T),y})$$

which, when combined with (6.3) and (6.6), completes the proof. $\qquad\square$

For the remainder of §6, suppose that $T \in \Sigma(\alpha)$ is singular, and denote by t_1 and t_2 the diagonal entries of T. As $t_1 t_2$ is a square, there are relatively prime integers n_1 and n_2 satisfying

$$(6.7) \qquad\qquad t_1 = n_1^2 \cdot t \qquad t_2 = n_2^2 \cdot t$$

for some nonzero $t \in \mathbb{Z}$, uniquely determined by T. Each of n_1 and n_2 is uniquely determined up to sign, and (directly from the definition of $\Sigma(\alpha)$) these signs may be chosen so that

$$\alpha = (n_1 \varpi_1 + n_2 \varpi_2)^2 \cdot t.$$

This implies that the field extension $F(\sqrt{-\alpha})/\mathbb{Q}$ is biquadratic, $t > 0$, the field

$$K \stackrel{\text{def}}{=} \mathbb{Q}(\sqrt{-t})$$

is one of the two quadratic imaginary subfields of $F(\sqrt{-\alpha})$, and t is the largest integer with the property

$$\mathcal{O}_F[\sqrt{-\alpha}] \subset \mathcal{O}_F[\sqrt{-t}].$$

Furthermore [20, Lemma 6.4.1] provides an isomorphism of stacks

(6.8) $$\mathcal{Z}(t) \cong \mathcal{Z}(T),$$

which takes the triple $(\mathbf{A}_0, \lambda_0, s_0)$ to the quadruple $(\mathbf{A}_0, \lambda_0, n_1 s_0, n_2 s_0)$. Let m_0 be the ramification index of p in $K/\mathbb{Q}$, abbreviate $d_K = \operatorname{disc}(K/\mathbb{Q})$, and define $n \in \mathbb{Z}^+$ by $4t = -n^2 d_K$ so that n is the conductor of $\mathbb{Z}[\sqrt{-t}]$. Let

$$\chi = \left(\frac{d_K}{p}\right) \in \{-1, 0, 1\}.$$

Conjecture 6.3. *As above, let $T \in \Sigma(\alpha)$ be singular. For every $y \in Z(T)(\mathbb{F}_p^{\text{alg}})$*

(6.9) $$I_{\mathcal{O}_{Y_0,y}}([\mathcal{O}_Y]^\bullet, \mathcal{O}_{M_0}) + I_{\mathcal{O}_{Y_0,y}}([\mathcal{O}_Y]^{\text{ver}}, \mathcal{O}_{M_0}) = \frac{1}{2} \cdot \Gamma_p(T) \cdot \operatorname{ord}_p\left(\frac{4\alpha\alpha^\sigma}{t}\right)$$

where

$$\Gamma_p(T) = m_0 \cdot \frac{p^{\operatorname{ord}_p(n)+1} - 1}{p - 1} - \chi m_0 \cdot \frac{p^{\operatorname{ord}_p(n)} - 1}{p - 1}.$$

The motivation for the conjecture is simply that (6.9) is what is needed for the equality of Proposition 6.7 below, and hence also the main result Theorem 8.2, to hold without unwanted hypotheses. In what follows we will prove Conjecture 6.3 in many cases; *e.g.* if p is split in F, or if p is odd and inert in F. These proofs make essential use of the calculations of the companion paper [11].

Keep T and y as in Conjecture 6.3. Writing X for any one of Y, Y_0, M, or M_0, and viewing y as a geometric point of X, abbreviate R_X for the completed strictly Henselian local ring of X at y. As in the proof of Proposition 6.2, the local ring $\mathcal{O}_{Y,y}$ is Cohen-Macaulay of dimension one, and hence the same is true of R_Y. In particular $R_Y/\mathfrak{p}$ has dimension one for every minimal prime $\mathfrak{p}$ of R_Y. Mimicking Definition 6.1, a minimal prime $\mathfrak{p}$ of R_Y is *improper* if it lies in the image of $\operatorname{Spec}(R_{Y_0}) \to \operatorname{Spec}(R_Y)$. We say that $\mathfrak{p}$ is *proper* otherwise, and abbreviate $\Pi^{\text{prop}}(R_Y)$ for the set of proper minimal primes of R_Y. For a minimal prime $\mathfrak{p} \subset R_Y$ abbreviate

$$\operatorname{mult}(\mathfrak{p}) = \operatorname{length}_{R_{Y,\mathfrak{p}}}(R_{Y,\mathfrak{p}}).$$

Routine commutative algebra shows that the left hand side of (6.9) can be computed after passing from $\mathcal{O}_{Y,y}$ to R_Y:

$$I_{\mathcal{O}_{Y_0,y}}([\mathcal{O}_Y]^{\text{prop}}, \mathcal{O}_{M_0})$$
$$= \sum_{\mathfrak{p} \in \Pi^{\text{prop}}(R_Y)} \operatorname{mult}(\mathfrak{p}) \cdot \sum_{\ell \geq 0} (-1)^\ell \operatorname{length}_{R_{Y_0}} \operatorname{Tor}_\ell^{R_M}(R_Y/\mathfrak{p}, R_{M_0}).$$

The argument leading to (2.5) shows that only the $\ell = 0$ term contributes to the inner sum, and thus

(6.10) $$I_{\mathcal{O}_{Y_0,y}}([\mathcal{O}_Y]^{\text{prop}}, \mathcal{O}_{M_0}) = \sum_{\mathfrak{p} \in \Pi^{\text{prop}}(R_Y)} \operatorname{mult}(\mathfrak{p}) \cdot \operatorname{length}_{R_{Y_0}}(R_Y/\mathfrak{p} \otimes_{R_M} R_{M_0}).$$

Keeping the notation above, let $W = W(\mathbb{F}_p^{\mathrm{alg}})$ be the ring of Witt vectors of $\mathbb{F}_p^{\mathrm{alg}}$, and let **Art** be the category of local Artinian W-algebras with residue field $\mathbb{F}_p^{\mathrm{alg}}$. Using (6.1) and (6.8), the point $y \in Z(T)(\mathbb{F}_p^{\mathrm{alg}})$ determines triples

$$(\mathbf{A}_0, \lambda_0, s_0) \in \mathcal{Z}(t)(\mathbb{F}_p^{\mathrm{alg}}) \qquad (\mathbf{A}, \lambda, t_\alpha) \in \mathcal{Y}(\mathbb{F}_p^{\mathrm{alg}})$$

related by $\mathbf{A} \cong \mathbf{A}_0 \otimes \mathcal{O}_F$ and

$$t_\alpha = s_0 \otimes (n_1 \varpi_1 + n_2 \varpi_2) \in \mathrm{End}(\mathbf{A}_0) \otimes_{\mathbb{Z}} \mathcal{O}_F \cong \mathrm{End}(\mathbf{A}).$$

Let $\mathbf{A}_{0,p}$ and $\mathbf{A}_p$ be the p-Barsotti-Tate groups of $\mathbf{A}_0$ and $\mathbf{A}$. The action of

$$\mathcal{O}_{B_0,p} \cong M_2(\mathbb{Z}_p)$$

(for any $\mathbb{Z}$-module S we abbreviate $S_p = S \otimes_{\mathbb{Z}} \mathbb{Z}_p$) allows us to decompose

$$\mathbf{A}_{0,p} \cong \mathfrak{g}_0 \times \mathfrak{g}_0$$

with $\mathfrak{g}_0$ a p-Barsotti-Tate group of dimension one and height two equipped with an action of the quadratic $\mathbb{Z}_p$-order $\mathbb{Z}_p[s_0] \cong \mathbb{Z}_p[x]/(x^2 + t)$. Similarly

$$\mathbf{A}_p \cong \mathfrak{g} \times \mathfrak{g}$$

where $\mathfrak{g} \cong \mathfrak{g}_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_{F,p}$ is equipped with an action of the quadratic $\mathcal{O}_{F,p}$-order $\mathcal{O}_{F,p}[t_\alpha] \cong \mathcal{O}_{F,p}[x]/(x^2 + \alpha)$. The quotient map $M \to \mathcal{M}_{/\mathbb{Z}_p}$ is étale, and so R_M is isomorphic to the completion of the strictly Henselian local ring of $\mathcal{M}$ at y. It follows from the Serre-Tate theory that the formal W-scheme $\mathrm{Spf}(R_M)$ classifies deformations of $\mathfrak{g}$ with its $\mathcal{O}_{F,p}$-action to objects of **Art** (the polarization λ lifts uniquely to any deformation of $\mathbf{A}$ by [2, p. 51] or [27]). Similarly R_Y classifies deformations of $\mathfrak{g}$ with its $\mathcal{O}_{F,p}[t_\alpha]$-action, R_{M_0} classifies deformations of $\mathfrak{g}_0$, and R_{Y_0} classifies deformations of $\mathfrak{g}_0$ with its $\mathbb{Z}_p[s_0]$-action.

Note that $\mathfrak{g}_0$ is either isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p \times \mu_{p^\infty}$ (the *ordinary case*) or to the unique connected p-Barsotti-Tate group of dimension one and height two (the *supersingular case*). The endomorphism s_0 of $\mathfrak{g}_0$ induces an embedding of $K_p \cong \mathbb{Q}_p[s_0]$ into $\mathrm{End}(\mathfrak{g}_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, from which we see that $\chi = 1$ implies that $\mathfrak{g}_0$ is ordinary, while $\chi \neq 1$ implies that $\mathfrak{g}_0$ is supersingular.

Proposition 6.4. *If $T \in \Sigma(\alpha)$ is singular and $\chi = 1$ then (6.9) holds for every $y \in Z(T)(\mathbb{F}_p^{\mathrm{alg}})$.*

Proof. If we define rank one $\mathbb{Z}_p$-modules

$$P_0 = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \mathfrak{g}_0) \qquad P_0^\vee = \mathrm{Hom}(\mathfrak{g}_0, \mu_{p^\infty})$$

and $\mathbf{P}_0 = P_0 \otimes_{\mathbb{Z}_p} P_0^\vee$, then the theory of Serre-Tate coordinates as in [8, Theorem 7.2] provides a canonical isomorphism of functors on **Art**

$$\mathrm{Spf}(R_{M_0}) \cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{P}_0, \widehat{\mathbb{G}}_m).$$

If S is an object of **Art** and $\phi : \mathbf{P}_0 \to \widehat{\mathbb{G}}_m(S)$ represents a deformation of $\mathfrak{g}_0$ to S, the endomorphism s_0 of $\mathfrak{g}_0$ lifts to this deformation if and only if ϕ satisfies

$$\phi((s_0 x) \otimes y) = \phi(x \otimes (s_0 y))$$

for every $x \in P_0$ and $y \in P_0^\vee$. It follows that there is an isomorphism of functors on **Art**

$$\mathrm{Spf}(R_{Y_0}) \cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{P}_0/\mathfrak{c}_0 \mathbf{P}_0, \widehat{\mathbb{G}}_m)$$

where $\mathfrak{c}_0 = n\mathbb{Z}_p$ is the conductor of the order $\mathbb{Z}_p[s_0]$. If we define rank one $\mathcal{O}_{F,p}$-modules

$$P = \operatorname{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \mathfrak{g}) \qquad P^\vee = \operatorname{Hom}(\mathfrak{g}, \mu_{p^\infty})$$

and $\mathbf{P} = P \otimes_{\mathcal{O}_{F,p}} P^\vee$, then similarly there are isomorphisms

$$\operatorname{Spf}(R_M) \cong \operatorname{Hom}_{\mathbb{Z}_p}(\mathbf{P}, \widehat{\mathbb{G}}_m)$$

and

$$\operatorname{Spf}(R_Y) \cong \operatorname{Hom}_{\mathbb{Z}_p}(\mathbf{P}/\mathfrak{c}\mathbf{P}, \widehat{\mathbb{G}}_m)$$

where $\mathfrak{c} \subset \mathcal{O}_{F,p}$ is the conductor of $\mathcal{O}_{F,p}[t_\alpha]$. Note that there are canonical isomorphisms

$$P \cong P_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_{F,p} \qquad P^\vee \cong P_0^\vee \otimes_{\mathbb{Z}_p} \omega$$

and $\mathbf{P} \cong \mathbf{P}_0 \otimes_{\mathbb{Z}_p} \omega$, where $\omega = \operatorname{Hom}_{\mathbb{Z}_p}(\mathcal{O}_{F,p}, \mathbb{Z}_p)$. After fixing an isomorphism $\mathbf{P}_0 \cong \mathbb{Z}_p$, the commutative diagram of functors on **Art**

(6.11)
$$\begin{array}{ccc} \operatorname{Spf}(R_{Y_0}) & \longrightarrow & \operatorname{Spf}(R_Y) \\ \downarrow & & \downarrow \\ \operatorname{Spf}(R_{M_0}) & \longrightarrow & \operatorname{Spf}(R_M) \end{array}$$

becomes identified with

(6.12)
$$\begin{array}{ccc} \operatorname{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p/\mathfrak{c}_0, \widehat{\mathbb{G}}_m) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}_p}(\omega/\mathfrak{c}\omega, \widehat{\mathbb{G}}_m) \\ \downarrow & & \downarrow \\ \operatorname{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p, \widehat{\mathbb{G}}_m) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}_p}(\omega, \widehat{\mathbb{G}}_m). \end{array}$$

Here the horizontal arrows are obtained by dualizing the $\mathbb{Z}_p$-module map $\operatorname{Tr} : \omega \to \mathbb{Z}_p$ defined by $\operatorname{Tr}(f) = f(1)$.

Set $c_0 = \operatorname{ord}_p(n)$, so that $\mathfrak{c}_0 = p^{c_0}\mathbb{Z}_p$, and define nonnegative integers c_1, c_2 as follows:

   (a) if $F_p \cong \mathbb{Q}_p \times \mathbb{Q}_p$ then $(p^{c_1}, p^{c_2}) \in \mathbb{Z}_p \times \mathbb{Z}_p \cong \mathcal{O}_{F,p}$ generates the conductor of the order $\mathcal{O}_{F,p}[t_\alpha]$,
   (b) if F_p is an unramified field extension of $\mathbb{Q}_p$ then $c_1 = c_2 = c_0$,
   (c) if F_p is a ramified field extension of $\mathbb{Q}_p$ then let ϖ_F be a uniformizer of F_p and let ϖ_F^c generate the conductor of the order $\mathcal{O}_{F,p}[t_\alpha]$. If c is even define $c_1 = c_2 = c/2$; if c is odd define $c_1 = (c-1)/2$ and $c_2 = (c+1)/2$.

One can check that in all cases $c_0 = \min\{c_1, c_2\}$,

$$\operatorname{ord}_p(4\alpha\alpha^\sigma) - \operatorname{ord}_p(t) = 2c_1 + 2c_2 - 2c_0,$$

and ω admits a $\mathbb{Z}_p$-basis $\{e_1, e_2\}$ such that $\operatorname{Tr}(e_i) = 1$ and $\{p^{c_1}e_1, p^{c_2}e_2\}$ is a $\mathbb{Z}_p$-basis of $\mathfrak{c}\omega$. Using this basis of ω one identifies the diagram (6.12) with

$$\begin{array}{ccc} \mu_{p^{c_0}} & \longrightarrow & \mu_{p^{c_1}} \times \mu_{p^{c_2}} \\ \downarrow & & \downarrow \\ \widehat{\mathbb{G}}_m & \longrightarrow & \widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m \end{array}$$

where the horizontal arrows are the diagonal maps and the vertical arrows are the natural inclusions. The original diagram (6.11) is now identified with

$$\begin{array}{ccc} \mathrm{Spf}(W[[x_0]]/(f_{c_0}(x_0))) & \longrightarrow & \mathrm{Spf}(W[[x_1,x_2]]/(f_{c_1}(x_1),f_{c_2}(x_2))) \\ \downarrow & & \downarrow \\ \mathrm{Spf}(W[[x_0]]) & \longrightarrow & \mathrm{Spf}(W[[x_1,x_2]]) \end{array}$$

where

$$f_c(x) = (x+1)^{p^c} - 1$$

and the horizontal arrows are determined by $x_i \mapsto x_0$.

The calculation of the right hand side of (6.10) is now reduced to a pleasant exercise. Let M be the fraction field of W and fix an embedding $M \to \mathbb{C}_p$. For each nonnegative integer k set $\varphi_k(x) = \Phi_{p^k}(x+1)$ where Φ_{p^k} is the p^k-cyclotomic polynomial, let X_k denote the roots of $\varphi_k(x)$ in $\mathbb{C}_p$, set $M_k = M(X_k)$, and let W_k be the ring of integers of M_k. The minimal primes of

$$\begin{aligned} R_Y & \cong W[[x_1,x_2]]/(f_{c_1}(x_1),f_{c_2}(x_2)) \\ & \cong W[x_1]/(f_{c_1}) \otimes_W W[x_2]/(f_{c_2}) \end{aligned}$$

are indexed by the $\mathrm{Aut}(\mathbb{C}_p/M)$-orbits of the set

$$(\mu_{p^{c_1}} \times \mu_{p^{c_2}})(\mathbb{C}_p) = \bigsqcup_{\substack{0 \le k_1 \le c_1 \\ 0 \le k_2 \le c_2}} (X_{k_1} \times X_{k_2})$$

by the rule that attaches to the orbit $[\pi_1, \pi_2]$ of the pair $(\pi_1, \pi_2) \in X_{k_1} \times X_{k_2}$ the kernel $\mathfrak{p}$ of the unique W-algebra homomorphism $R_Y \to \mathbb{C}_p$ taking $x_i \mapsto \pi_i$. Assuming for simplicity that $k_1 \le k_2$, the localization of R_Y at $\mathfrak{p}$ is isomorphic to the cyclotomic field M_{k_2}, and so $\mathrm{mult}(\mathfrak{p}) = 1$. Under the above indexing the proper minimal primes of R_Y correspond to those orbits of the form $[\pi_1, \pi_2]$ with $\pi_1 \neq \pi_2$.

If $\mathfrak{p} \in \Pi^{\mathrm{prop}}(R_Y)$ is indexed by the orbit $[\pi_1, \pi_2] \subset X_{k_1} \times X_{k_2}$ then we will say that $\mathfrak{p}$ has *type* (k_1, k_2). Let $\Pi^{\mathrm{prop}}_{k_1, k_1}(R_Y) \subset \Pi^{\mathrm{prop}}(R_Y)$ be the subset of components having type (k_1, k_2). Suppose $k_1 < k_2$, fix some $\mathfrak{p} \in \Pi^{\mathrm{prop}}_{k_1, k_2}(R_Y)$, and let $[\pi_1, \pi_2]$ be the corresponding orbit. There is an isomorphism $R_Y/\mathfrak{p} \cong W_{k_2}$ defined by $x_i \mapsto \pi_i$ and isomorphisms

$$R_Y/\mathfrak{p} \otimes_{R_M} R_{M_0} \cong W_{k_2}/(\pi_2 - \pi_1) \cong \mathbb{F}_p^{\mathrm{alg}}.$$

The second isomorphism is due to the fact that $\pi_2 - \pi_1$ is a uniformizing parameter of W_{k_2}. It is easy to see that the number of $\mathrm{Aut}(\mathbb{C}_p/M)$-orbits in $X_{k_1} \times X_{k_2}$ is $|X_{k_1}| = [M_{k_1} : M]$, and applying the same reasoning in the case $k_2 < k_1$ shows that for any $k_1 \neq k_2$ we have

$$\sum_{\mathfrak{p} \in \Pi^{\mathrm{prop}}_{k_1,k_2}(R_Y)} \mathrm{mult}(\mathfrak{p}) \cdot \mathrm{length}_{R_Y}(R_Y/\mathfrak{p} \otimes_W R_{M_0}) = |\Pi^{\mathrm{prop}}_{k_1,k_2}(R_Y)| = [M_{\min\{k_1,k_2\}} : M].$$

Next fix $0 \le k \le c_0$ and one element $\pi \in X_k$. The minimal primes of R_Y contained in $\Pi^{\mathrm{prop}}_{k,k}(R_Y)$ correspond to the orbits $[\pi, \pi']$ as π' ranges over $X_k \smallsetminus \{\pi\}$,

and we find

$$\sum_{\mathfrak{p}\in\Pi_{k,k}^{\mathrm{prop}}(R_Y)} \mathrm{mult}(\mathfrak{p})\cdot\mathrm{length}_{R_Y}(R_Y/\mathfrak{p}\otimes_W R_{M_0})$$

$$= \sum_{\substack{\pi'\in X_k \\ \pi'\neq\pi}} \mathrm{length}_{W_k}\big(W_k/(\pi-\pi')\big)$$

$$= \mathrm{ord}_\pi(\mathrm{Diff}(M_k/M))$$

where $\mathrm{Diff}(M_k/M)$ is the relative different. Combining (6.10) with the equalities

$$[M_k : M] = \begin{cases} 1 & \text{if } k=0 \\ p^{k-1}(p-1) & \text{if } k>0 \end{cases}$$

and (denoting by ϖ_k any uniformizer of M_k and using [20, Proposition 7.8.5])

$$\mathrm{ord}_{\varpi_k}(\mathrm{Diff}(M_k/M)) = \begin{cases} 0 & \text{if } k=0 \\ p^{k-1}(kp-k-1) & \text{if } k>0 \end{cases}$$

an elementary calculation gives

$$\begin{aligned}
I_{\mathcal{O}_{Y_0,y}}([\mathcal{O}_Y]^{\mathrm{prop}},\mathcal{O}_{M_0}) &= \sum_{\mathfrak{p}\in\Pi^{\mathrm{prop}}(R_Y)} \mathrm{mult}(\mathfrak{p})\cdot\mathrm{length}_{R_Y}(R_Y/\mathfrak{p}\otimes_W R_{M_0}) \\
&= \sum_{\substack{0\leq k_i\leq c_i \\ k_1\neq k_2}} [M_{\min\{k_1,k_2\}} : M] + \sum_{0\leq k\leq c_0} \mathrm{ord}_{\varpi_k}(\mathrm{Diff}(M_k/M)) \\
&= p^{c_0}\cdot(c_1+c_2-c_0) \\
&= \frac{1}{2}\cdot p^{\mathrm{ord}_p(n)}\cdot\mathrm{ord}_p\left(\frac{4\alpha\alpha^\sigma}{t}\right)
\end{aligned}$$

as claimed. □

Proposition 6.5. *If $T\in\Sigma(\alpha)$ is singular and p splits in F then (6.9) holds for every $y\in Z(T)(\mathbb{F}_p^{\mathrm{alg}})$.*

Proof. After Proposition 6.4 we may assume that $\chi\neq 1$, so that $K_p/\mathbb{Q}_p$ is a quadratic field extension, and we are in the supersingular case. As $\mathrm{End}(\mathfrak{g}_0)$ is the maximal order in a nonsplit quaternion algebra over $\mathbb{Q}_p$, the embedding

$$\mathbb{Z}_p[x]/(x^2+t)\to\mathrm{End}(\mathfrak{g}_0)$$

determined by s_0 extends to an embedding $\mathcal{O}_{K,p}\to\mathrm{End}(\mathfrak{g}_0)$, and the action of $\mathcal{O}_{K,p}$ on $\mathrm{Lie}(\mathfrak{g}_0)$ is through a $\mathbb{Z}_p$-algebra homomorphism $\mathcal{O}_{K,p}\to\mathbb{F}_p^{\mathrm{alg}}$. If we let W_0 be the completion of the strict Henselization of $\mathcal{O}_{K,p}$ with respect to this map, then W_0 is naturally a W-algebra satisfying

$$W_0\cong\begin{cases} W & \text{if } \chi=-1 \\ \mathcal{O}_{K,p}\otimes_{\mathbb{Z}_p} W & \text{if } \chi=0. \end{cases}$$

The field K_p is naturally a subfield of $M_0=\mathrm{Frac}(W_0)$, and by local class field theory $\mathcal{O}_{K,p}^\times$ is isomorphic to $\mathrm{Gal}(\widehat{K}^{\mathrm{ab}}/M_0)$, where $\widehat{K}^{\mathrm{ab}}$ is the completion of the maximal abelian extension of K. Let $M_0\subset M_k\subset\widehat{K}^{\mathrm{ab}}$ be the subextension characterized by

$$\mathcal{O}_{K,p}^\times/(\mathbb{Z}_p+p^k\mathcal{O}_{K,p})^\times\cong\mathrm{Gal}(M_k/M_0)$$

and let W_k be the integer ring of M_k. As in [11, §4.1] there is an isomorphism of W-algebras $R_{M_0} \cong W[[x]]$, while the Gross-Keating theory of quasi-canonical lifts implies that

$$R_{Y_0} \cong W[[x]]/(f_{c_0}(x))$$

where $c_0 = \mathrm{ord}_p(n)$ (so that p^{c_0} generates the conductor of the quadratic $\mathbb{Z}_p$-order $\mathbb{Z}_p[s_0]$) and

$$f_{c_0}(x) = \prod_{k=0}^{c_0} \varphi_k(x)$$

with each $\varphi_k(x)$ an Eisenstein polynomial satisfying $W[[x]]/(\varphi_k(x)) \cong W_k$.

Fix an isomorphism $\mathcal{O}_{F,p} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and let $(p^{c_1}, p^{c_2}) \in \mathcal{O}_{F,p}$ generate the conductor of the quadratic $\mathcal{O}_{F,p}$-order $\mathcal{O}_{F,p}[t_\alpha]$. As in the proof of Proposition 6.4, we have $c_0 = \min\{c_1, c_2\}$ and

$$\mathrm{ord}_p\left(\frac{4\alpha\alpha^\sigma}{t}\right) = 2c_1 + 2c_2 - 2c_0 + \mathrm{ord}_p(d_K).$$

The induced splitting $\mathfrak{g} \cong \mathfrak{g}_0 \times \mathfrak{g}_0$ determines isomorphisms $R_M \cong R_{M_0} \widehat{\otimes}_W R_{M_0}$ and

$$R_Y \cong W[[x_1]]/(f_{c_1}(x_1)) \widehat{\otimes}_W W[[x_2]]/(f_{c_2}(x_2)),$$

and so the commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spf}(R_{Y_0}) & \longrightarrow & \mathrm{Spf}(R_Y) \\
\downarrow & & \downarrow \\
\mathrm{Spf}(R_{M_0}) & \longrightarrow & \mathrm{Spf}(R_M)
\end{array}
$$

of functors on **Art** can be identified with the commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spf}(W[[x_0]]/(f_{c_0}(x_0))) & \longrightarrow & \mathrm{Spf}(W[[x_1, x_2]]/(f_{c_1}(x_1), f_{c_2}(x_2))) \\
\downarrow & & \downarrow \\
\mathrm{Spf}(W[[x_0]]) & \longrightarrow & \mathrm{Spf}(W[[x_1, x_2]])
\end{array}
$$

in which the horizontal arrows are determined by $x_i \mapsto x_0$. Imitating the proof of Proposition 6.5 shows that

$$
\begin{aligned}
I_{\mathcal{O}_{Y_0},y}([\mathcal{O}_Y]^{\mathrm{prop}}, \mathcal{O}_{M_0}) &= \sum_{\mathfrak{p} \in \Pi^{\mathrm{prop}}(R_Y)} \mathrm{mult}(\mathfrak{p}) \cdot \mathrm{length}_{R_Y}(R_Y/\mathfrak{p} \otimes_W R_{M_0}) \\
&= \sum_{\substack{0 \le k_i \le c_i \\ k_1 \ne k_2}} [M_{\min\{k_1, k_2\}} : M] + \sum_{0 \le k \le c_0} \mathrm{ord}_{\varpi_k}(\mathrm{Diff}(M_k/M))
\end{aligned}
$$

where in the final sum ϖ_k is a uniformizer of M_k. The final sum can be computed using the formulas of [20, Proposition 7.7.7] and [20, Proposition 7.8.5]. If $\chi = -1$ then for all $k > 0$

$$[M_k : M] = p^{k-1}(p+1)$$

and

$$\mathrm{ord}_{\varpi_k}(\mathrm{Diff}(M_k/M)) = kp^{k-1}(p+1) - \frac{p^k + p^{k-1} - 2}{p-1}.$$

If instead $\chi = 0$, then for all $k \geq 0$ we have $[M_k : M] = 2p^k$ and

$$\mathrm{ord}_{\varpi_k}(\mathrm{Diff}(M_k/M)) = 2kp^k - 2\frac{p^k - 1}{p - 1} + p^k \cdot \mathrm{ord}_p(d_K).$$

Tedious but elementary calculation then results in

$$
\begin{aligned}
I_{\mathcal{O}_{Y_0,y}}&([\mathcal{O}_Y]^{\mathrm{prop}}, \mathcal{O}_{M_0}) \\
&= m_0 \left(\frac{p^{c_0+1} - 1}{p - 1} - \chi \frac{p^{c_0} - 1}{p - 1} \right) \cdot \left(c_1 + c_2 - c_0 + \frac{\mathrm{ord}_p(d_K)}{2} \right) \\
&= \frac{m_0}{2} \cdot \left(\frac{p^{c_0+1} - 1}{p - 1} - \chi \frac{p^{c_0} - 1}{p - 1} \right) \cdot \mathrm{ord}_p \left(\frac{4\alpha\alpha^\sigma}{t} \right).
\end{aligned}
$$

$\square$

We are left to verify (6.9) in the supersingular case with p nonsplit in F. This is much harder than the cases treated in Propositions 6.4 and 6.5, and the bulk of the proof is contained in the separate article [11].

Proposition 6.6. *Suppose that $T \in \Sigma(\alpha)$ is singular, that p is odd, and that $p\mathcal{O}_F$ is relatively prime to $\gcd(\alpha\mathcal{O}_F, \mathfrak{D}_F)$. Then (6.9) holds for every $y \in Z(T)(\mathbb{F}_p^{\mathrm{alg}})$.*

Proof. After Propositions 6.4 and 6.5, we may assume that p is nonsplit in F and that $\chi \neq 1$ (so that we are in the supersingular case). Set $c_0 = \mathrm{ord}_p(n)$.

First assume that p is inert in F. Combining (6.10) with [11, Theorem D] (where K_p is denoted E_0) gives

$$I_{\mathcal{O}_{Y_0},y}([\mathcal{O}_Y]^{\mathrm{prop}}, \mathcal{O}_{M_0}) = c_0 \left(\frac{p^{c_0+1} - 1}{p - 1} + \frac{p^{c_0} - 1}{p - 1} \right)$$

if $\chi = -1$, and

$$I_{\mathcal{O}_{Y_0},y}([\mathcal{O}_Y]^{\mathrm{prop}}, \mathcal{O}_{M_0}) = (2c_0 + 1) \frac{p^{c_0+1} - 1}{p - 1}$$

if $\chi = 0$. If we set $\mathfrak{p} = p\mathcal{O}_F$ then

$$\mathrm{ord}_{\mathfrak{p}}(\alpha) = \mathrm{ord}_{\mathfrak{p}}(\alpha^\sigma) = \mathrm{ord}_p(t)$$

which, when combined with $4t = -n^2 d_K$ and $p \neq 2$, implies that

$$\mathrm{ord}_p \left(\frac{4\alpha\alpha^\sigma}{t} \right) = 2c_0 + (\chi + 1).$$

Thus (6.9) holds.

We are left with the case of p ramified in F. Write $p\mathcal{O}_F = \mathfrak{p}^2$ and note that our hypotheses on p imply that $\mathfrak{p}$ is relatively prime to $\alpha\mathcal{O}_F$. It follows that t is relatively prime to p, and hence from $4t = -n^2 d_K$ that $c_0 = 0$ and that K is unramified at p. In particular the right hand side of (6.9) is equal to 0. We may now invoke [11, Proposition 5.1.1], which asserts that $R_{Y_0} \cong R_Y \cong W$. Therefore $\Pi^{\mathrm{prop}}(R_Y) = \emptyset$ and the right hand side of (6.10) (and so also the left hand side of (6.9)) is equal to 0. $\square$

Define a codimension two cycle on M by

$$(6.13) \qquad\qquad C_p^\bullet = \sum_{D \in \Pi^\bullet(Y)} \mathrm{mult}_D(\mathcal{O}_Y) \cdot \phi(D),$$

where $\phi(D)$ is viewed as a closed subscheme of M with its reduced subscheme structure. Define $C_p^{\bullet\bullet}$ and C_p^{ver} in the same way, replacing $\Pi^\bullet(Y)$ with $\Pi^{\bullet\bullet}(Y)$ or $\Pi^{\mathrm{ver}}(Y)$, respectively. Each of $C_p^\bullet$, $C_p^{\bullet\bullet}$, or C_p^{ver} is H-invariant and so (by [6, Lemma 4.2]) determines a cycle on $\mathcal{M}_{/\mathbb{Z}_p}$, which we denote by $\mathcal{C}_p^\bullet$, $\mathcal{C}_p^{\bullet\bullet}$, or $\mathcal{C}_p^{\mathrm{ver}}$. The sum $\mathcal{C}_p^\bullet + \mathcal{C}_p^{\bullet\bullet} + \mathcal{C}_p^{\mathrm{ver}}$ represents the cycle class $\mathcal{C}_p \in \mathrm{CH}^2_{\mathcal{Y}_{/\mathbb{Z}_p}}(\mathcal{M}_{/\mathbb{Z}_p})$ constructed in [12, §3.3].

Proposition 6.7. *Assume that at least one of the following hypotheses holds:*

(a) $F(\sqrt{-\alpha})/\mathbb{Q}$ *is not biquadratic,*

(b) p *splits in* F,

(c) p *is odd and* $p\mathcal{O}_F$ *is relatively prime to* $\gcd(\alpha\mathcal{O}_F, \mathfrak{D}_F)$.

Then

$$I_p(\mathcal{C}_p^\bullet, \mathcal{M}_0) + I_p(\mathcal{C}_p^{\mathrm{ver}}, \mathcal{M}_0) = \frac{1}{2} \sum_{\substack{T \in \Sigma(\alpha) \\ \det(T)=0}} \deg_{\mathbb{Q}}(\mathcal{Z}(t)) \cdot \mathrm{ord}_p\left(\frac{4\alpha\alpha^\sigma}{t}\right)$$

$$+ \sum_{\substack{T \in \Sigma(\alpha) \\ \det(T)\neq 0}} \sum_{y \in \mathcal{Z}(T)(\mathbb{F}_p^{\mathrm{alg}})} e_y^{-1} \cdot \mathrm{length}_{\mathcal{O}^{\mathrm{sh}}_{\mathcal{Z}(T),y}}(\mathcal{O}^{\mathrm{sh}}_{\mathcal{Z}(T),y}).$$

On the right hand side t is defined by (6.7) and $\mathcal{O}^{\mathrm{sh}}_{\mathcal{Z}(T),y}$ is the strictly Henselian local ring of $\mathcal{Z}(T)$ at y. The rational number $\deg_{\mathbb{Q}}(\mathcal{D})$ is defined by (5.11) for an irreducible cycle $\mathcal{D}$ of codimension two on $\mathcal{M}$, and extended linearly to all codimension two cycles.

Proof. If (a) holds than $\Sigma(\alpha)$ contains no singular matrices by [12, Lemma 3.1.5(c)], and so (6.1) and Proposition 6.2 imply

$$I_p(C_p^\bullet, M_0) + I_p(C_p^{\mathrm{ver}}, M_0)$$

$$= \sum_{y \in Y_0(\mathbb{F}_p^{\mathrm{alg}})} \left(I_{\mathcal{O}_{Y_0,y}}([\mathcal{O}_Y]^\bullet, \mathcal{O}_{M_0}) + I_{\mathcal{O}_{Y_0,y}}([\mathcal{O}_Y]^{\mathrm{ver}}, \mathcal{O}_{M_0})\right)$$

$$= \sum_{T \in \Sigma(\alpha)} \sum_{y \in Z(T)(\mathbb{F}_p^{\mathrm{alg}})} \mathrm{length}_{\mathcal{O}_{Z(T),y}}(\mathcal{O}_{Z(T),y}).$$

If (b) holds then combining (6.1) with Propositions 6.2 and 6.5 gives

$$I_p(C_p^\bullet, M_0) + I_p(C_p^{\mathrm{ver}}, M_0)$$

$$= \sum_{y \in Y_0(\mathbb{F}_p^{\mathrm{alg}})} \left(I_{\mathcal{O}_{Y_0,y}}([\mathcal{O}_Y]^\bullet, \mathcal{O}_{M_0}) + I_{\mathcal{O}_{Y_0,y}}([\mathcal{O}_Y]^{\mathrm{ver}}, \mathcal{O}_{M_0})\right)$$

$$= \sum_{\substack{T \in \Sigma(\alpha) \\ \det(T)\neq 0}} \sum_{y \in Z(T)(\mathbb{F}_p^{\mathrm{alg}})} \mathrm{length}_{\mathcal{O}_{Z(T),y}}(\mathcal{O}_{Z(T),y})$$

$$+ \sum_{\substack{T \in \Sigma(\alpha) \\ \det(T)=0}} \frac{1}{2} \cdot \Gamma_p(T) \cdot \mathrm{ord}_p\left(\frac{4\alpha\alpha^\sigma}{t}\right) \cdot |Z(T)(\mathbb{F}_p^{\mathrm{alg}})|.$$

If (c) holds then one obtains the same equalities by replacing Proposition 6.5 with Proposition 6.6. In all cases the desired result follows by dividing the above equalities by $|H|$ and using the equality

$$|Z(t)(\mathbb{Q}^{\mathrm{alg}})| = \Gamma_p(T) \cdot |Z(T)(\mathbb{F}_p^{\mathrm{alg}})|$$

of [20, Proposition 7.7.7(ii)] for each singular $T \in \Sigma(\alpha)$.   □

7. Ramified intersection theory

We continue with the work of the previous section, but now work in characteristic dividing the discriminant of B_0. The situation is complicated by the fact that $\mathcal{Y}(\alpha)$ may have vertical components of dimension 2, which must be removed and replaced, following the constructions of [12], by new vertical components of dimension 1. We intersect $\mathcal{M}_0$ against these new components, and against those horizontal components of $\mathcal{Y}(\alpha)$ that meet $\mathcal{M}_0$ properly. Our calculations rely heavily on the earlier work of Kudla-Rapoport-Yang [18, 19].

As in §6, fix a totally positive $\alpha \in \mathcal{O}_F$. Fix a prime p that divides $\mathrm{disc}(B_0)$ and recall from the introduction our hypothesis that all such primes are split in F. Let $W = W(\mathbb{F}_p^{\mathrm{alg}})$ be the ring of Witt vectors of $\mathbb{F}_p^{\mathrm{alg}}$, let $\mathbb{A}_f$ be the ring of finite adeles of $\mathbb{Q}$, and let $\mathbb{A}_f^p$ be the prime-to-p part of $\mathbb{A}_f$. Define compact open subgroups of $G_0(\mathbb{A}_f)$ and $G(\mathbb{A}_f)$ by

$$U_0^{\mathrm{max}} = \widehat{\mathcal{O}}_{B_0}^\times \qquad U^{\mathrm{max}} = \{b \in \widehat{\mathcal{O}}_B^\times : \mathrm{Nm}(b) \in \widehat{\mathbb{Z}}^\times\},$$

and choose a normal compact open subgroup $U \subset U^{\mathrm{max}}$ of the form $U = U_p U^p$ with $U_p \subset G(\mathbb{Q}_p)$ and $U^p \subset G(\mathbb{A}_f^p)$. We assume $U_p = U_p^{\mathrm{max}}$. For sufficiently small such U there is an isomorphism of DM stacks $\mathcal{M}_{/\mathbb{Z}_p} \cong [H \backslash M]$, where $H = U^{\mathrm{max}}/U$, and M is the $\mathbb{Z}_p$-scheme representing the functor that assigns to a $\mathbb{Z}_p$-scheme S the set of isomorphism classes of $\mathfrak{D}_F^{-1}$-polarized QM abelian fourfolds over S equipped with a U-level structure in the sense of [12, §3.1]. Having chosen such a presentation of $\mathcal{M}_{/\mathbb{Z}_p}$ let M_0, Y_0, Y, $Z(t)$, and $Z(T)$ have the same meaning as in §6.

Recall that for a Noetherian scheme X, we let $\mathbf{K}_0(X)$ be the Grothendieck group of the category of coherent $\mathcal{O}_X$-modules, and that the class of such a coherent $\mathcal{F}$ in $\mathbf{K}_0(X)$ is denoted $[\mathcal{F}]$. We denote by $\mathbf{K}_0^{\mathrm{ver}}(X)$ the Grothendieck group of the category of locally $\mathbb{Z}_p$-torsion coherent $\mathcal{O}_X$-modules. As X is quasi-compact every locally $\mathbb{Z}_p$-torsion coherent $\mathcal{O}_X$-module $\mathcal{F}$ satisfies $p^n \mathcal{F} = 0$ for some sufficiently large n. As in §6 let $\Pi^\bullet(Y)$ (respectively $\Pi^{\bullet\bullet}(Y)$) be the set of horizontal components of Y that are not contained in Y_0 (respectively are contained in Y_0), and view each such component as a closed subscheme of Y with its reduced subscheme structure. Let $Y^\bullet \subset Y$ be the union of all $D \in \Pi^\bullet(Y)$ with its reduced subscheme structure and define $Y^{\bullet\bullet}$ similarly. Using the notation of §6 define classes in $\mathbf{K}_0(Y^\bullet)$ and $\mathbf{K}_0(Y^{\bullet\bullet})$ by

$$[\mathcal{O}_Y]^\bullet = \sum_{D \in \Pi^\bullet(Y)} [\mathcal{O}_Y]_D \qquad [\mathcal{O}_Y]^{\bullet\bullet} = \sum_{D \in \Pi^{\bullet\bullet}(Y)} [\mathcal{O}_Y]_D.$$

While the horizontal components of Y are all of dimension one, Y may have vertical components of dimension two. Thus while we may define codimension two cycles $\mathcal{C}_p^\bullet$ and $\mathcal{C}_p^{\bullet\bullet}$ on $\mathcal{M}_{/\mathbb{Z}_p}$ exactly as in (6.13), the construction of a codimension two cycle $\mathcal{C}_p^{\mathrm{ver}}$ will proceed by the roundabout construction of an auxiliary class $[\mathfrak{O}_Y]^{\mathrm{ver}} \in \mathbf{K}_0^{\mathrm{ver}}(Y)$ to serve as a substitute for the naive class $[\mathcal{O}_Y]^{\mathrm{ver}}$. To construct this class, recall some notation and constructions from [12, §4]. Fix a principally polarized QM abelian surface $(\mathbf{A}_0^*, \lambda_0^*)$ over $\mathbb{F}_p^{\mathrm{alg}}$, set

$$(\mathbf{A}^*, \lambda^*) = (\mathbf{A}_0^*, \lambda_0^*) \otimes \mathcal{O}_F,$$

and define totally definite quaternion algebras over $\mathbb{Q}$ and F, respectively,

$$\overline{B}_0 = \mathrm{End}^0(\mathbf{A}_0^*) \qquad \overline{B} = \mathrm{End}^0(\mathbf{A}^*)$$

so that $\overline{B}_0 \otimes_{\mathbb{Q}} F \cong \overline{B}$. Let $\overline{G}_0 \subset \overline{G}$ be the algebraic groups over $\mathbb{Q}$ defined in the same way as $G_0 \subset G$, but with B_0 and B replaced by $\overline{B}_0$ and $\overline{B}$. Let $\widehat{\Lambda}_0$ and $\widehat{\Lambda}$ be the profinite completions of $\mathcal{O}_{B_0}$ and $\mathcal{O}_B$, respectively, and let

$$\widehat{\Lambda}_0^p \cong \widehat{\mathcal{O}}_{B_0}^p \qquad \widehat{\Lambda}^p \cong \widehat{\mathcal{O}}_B^p$$

be their prime-to-p parts. As in [12, §4.1], fix an isomorphism of $\mathbb{A}_f^p$-modules

$$\nu_0^* : \widehat{\Lambda}_0^p \to \mathrm{Ta}^p(A_0^*)$$

where on the right Ta^p is the prime-to-p adelic Tate module of the underlying abelian variety A_0^* of $\mathbf{A}_0^*$. This isomorphism is assumed to respect the left $\mathcal{O}_{B_0}$ action on both sides, and to identify the Weil pairing on the right induced by λ_0 with the pairing ψ_0 on the left defined in [12, §3.1]. By tensoring with $\mathcal{O}_F$, the choice of ν_0 induces an isomorphism

$$\nu^* : \widehat{\Lambda}^p \to \mathrm{Ta}^p(A^*).$$

Each $g \in \overline{B}_0 \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ acts as a B_0-linear endomorphism of $\mathrm{Ta}^p(A_0^*) \otimes_{\mathbb{Z}} \mathbb{Q}$, and so also acts (using ν_0) as a B_0-linear endomorphism of $\widehat{\Lambda}_0^p \otimes_{\mathbb{Z}} \mathbb{Q}$. As the action of B_0 on $\widehat{\Lambda}_0^p \otimes_{\mathbb{Z}} \mathbb{Q}$ is by left multiplication, the endomorphism of $\widehat{\Lambda}_0^p \otimes_{\mathbb{Z}} \mathbb{Q}$ determined by g is given by right multiplication by some $\iota_0(g) \in B_0 \otimes_{\mathbb{Q}} \mathbb{A}_f$. In this way the choice of ν_0^* determines a bijection

$$\iota_0 : \overline{G}_0(\mathbb{A}_f^p) \to G_0(\mathbb{A}_f^p),$$

which satisfies $\iota_0(xy) = \iota_0(y)\iota_0(x)$. Similarly ν^* determines a bijection

$$\iota : \overline{G}(\mathbb{A}_f^p) \to G(\mathbb{A}_f^p)$$

satisfying $\iota(xy) = \iota(y)\iota(x)$. The induced bijection between subgroups of $\overline{G}_0(\mathbb{A}_f^p)$ and subgroups of $G_0(\mathbb{A}_f^p)$ is denoted $\overline{H}^p \leftrightarrow H^p$, and similarly with G_0 replaced by G.

Let $\mathfrak{G}_0^*$ denote the p-divisible group of $\mathbf{A}_0^*$ equipped with its action of $\mathcal{O}_{B_0} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We denote by $\mathfrak{h}_m$ Drinfeld's formal W-scheme representing the functor that assigns to every W-scheme S on which p is locally nilpotent the set $\mathfrak{h}_m(S)$ of isomorphism classes of pairs $(\mathfrak{G}_0, \rho_0)$, in which $\mathfrak{G}_0$ is a special formal $\mathcal{O}_{B_0} \otimes_{\mathbb{Z}} \mathbb{Z}_p$-module (in the sense of [4, §II.2]) of dimension two and height four over S and

$$\rho_0 : \mathfrak{G}_0^* \times_{\mathbb{F}_p^{\mathrm{alg}}} S_{/\mathbb{F}_p^{\mathrm{alg}}} \to \mathfrak{G}_0 \times_S S_{/\mathbb{F}_p^{\mathrm{alg}}}$$

is a height $2m$ quasi-isogeny of p-divisible groups over $S_{/\mathbb{F}_p^{\mathrm{alg}}}$ respecting the action of $\mathcal{O}_{B_0} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. The group $\overline{G}_0(\mathbb{Q}_p)$ acts on the formal W-scheme

$$\mathfrak{X}_0 = \bigsqcup_{m \in \mathbb{Z}} \mathfrak{h}_m$$

by

$$\gamma \cdot (\mathfrak{G}_0, \rho_0) = (\mathfrak{G}_0, \rho_0 \circ \gamma^{-1}).$$

As p splits in F, fix an isomorphism $F \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \mathbb{Q}_p \times \mathbb{Q}_p$. This, in turn, determines an isomorphism

$$\overline{G}(\mathbb{Q}_p) \cong \{(x, y) \in \overline{G}_0(\mathbb{Q}_p) \times \overline{G}_0(\mathbb{Q}_p) : \mathrm{Nm}(x) = \mathrm{Nm}(y)\}$$

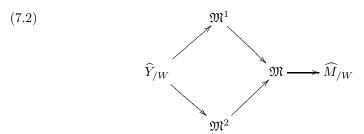and hence an action of $\overline{G}(\mathbb{Q}_p)$ on

$$\mathfrak{X} = \bigsqcup_{m \in \mathbb{Z}} (\mathfrak{h}_m \times_W \mathfrak{h}_m).$$

For any W-scheme X denote by $\widehat{X}$ the formal completion of X along its special fiber, a formal W-scheme.

There is a Čerednik-Drinfeld style isomorphism of formal W-schemes

$$(7.1) \qquad\qquad \widehat{M}_{/W} \cong \overline{G}(\mathbb{Q}) \backslash \mathfrak{X} \times \overline{G}(\mathbb{A}_f^p)/\overline{U}^p,$$

and from [12, §4] we have a commutative diagram of formal W-schemes

$$(7.2)$$

$$
\begin{array}{ccc}
 & \mathfrak{M}^1 & \\
\nearrow & & \searrow \\
\widehat{Y}_{/W} & \mathfrak{M} \longrightarrow \widehat{M}_{/W} & \\
\searrow & & \nearrow \\
 & \mathfrak{M}^2 &
\end{array}
$$

in which the square is cartesian and all arrows in the square are closed immersions. We quickly recall the definitions of $\mathfrak{M}$ and $\mathfrak{M}^k$. Let $\overline{V}$ denote the F-vector space of elements of $\overline{B}$ having reduced trace zero. We equip $\overline{V}$ with the F-valued quadratic form $\overline{Q}(\tau) = -\tau^2$ and let $\overline{G}(\mathbb{Q})$ act on $\overline{V}$ by conjugation. Similarly let $\overline{V}_0$ be the trace zero elements of $\overline{B}_0$ equipped with the conjugation action of $\overline{G}_0(\mathbb{Q})$ and with the quadratic form $\overline{Q}_0(\tau_0) = -\tau_0^2$. For each $\tau \in \overline{V} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ write (τ_1, τ_2) for the image of τ under

$$(7.3) \qquad\qquad \overline{V} \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong (\overline{V}_0 \otimes_{\mathbb{Q}} \mathbb{Q}_p) \times (\overline{V}_0 \otimes_{\mathbb{Q}} \mathbb{Q}_p).$$

For every W-scheme S on which p is locally nilpotent and every $\tau_0 \in \overline{V}_0 \otimes_{\mathbb{Q}} \mathbb{Q}_p$, viewed as a quasi-endomorphism of $\mathfrak{G}_0^* \times_{\mathbb{F}_p^{\mathrm{alg}}} S_{/\mathbb{F}_p^{\mathrm{alg}}}$, let

$$\mathfrak{h}_m(\tau_0)(S) \subset \mathfrak{h}_m(S)$$

be the subset consisting of those pairs $(\mathfrak{G}_0, \rho_0)$ for which the quasi-endomorphism

$$\rho_0 \circ \tau_0 \circ \rho_0^{-1} \in \mathrm{End}(\mathfrak{G}_0 \times_S S_{/\mathbb{F}_p^{\mathrm{alg}}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

lies in the image of

$$\mathrm{End}(\mathfrak{G}_0) \to \mathrm{End}(\mathfrak{G}_0 \times_S S_{/\mathbb{F}_p^{\mathrm{alg}}}).$$

The functor $\mathfrak{h}_m(\tau_0)$ is represented by a closed formal subscheme of $\mathfrak{h}_m$, and we define

$$\mathfrak{X}_0(\tau_0) = \bigsqcup_{m \in \mathbb{Z}} \mathfrak{h}_m(\tau_0).$$

For each $\tau \in \overline{V}$ define closed formal subschemes of $\mathfrak{X}$ by

$$\mathfrak{X}^1(\tau) = \bigsqcup_{m \in \mathbb{Z}} (\mathfrak{h}_m(\tau_1) \times_W \mathfrak{h}_m) \qquad \mathfrak{X}^2(\tau) = \bigsqcup_{m \in \mathbb{Z}} (\mathfrak{h}_m \times_W \mathfrak{h}_m(\tau_2))$$

and

$$\mathfrak{X}(\tau) = \mathfrak{X}^1(\tau) \times_{\mathfrak{X}} \mathfrak{X}^2(\tau).$$

Define a right $\overline{U}^p$-invariant compact open subset $\Omega(\tau) \subset \overline{G}(\mathbb{A}_f^p)$ by

$$\Omega(\tau) = \{g \in \overline{G}(\mathbb{A}_f^p) : \widehat{\Lambda}^p \cdot \iota(g^{-1}\tau g) \subset \widehat{\Lambda}^p\}.$$

Thus $\gamma \cdot \Omega(\tau) = \Omega(\gamma\tau\gamma^{-1})$ for each $\gamma \in \overline{G}(\mathbb{Q})$, and there is an isomorphism of formal W-schemes

(7.4) $$\widehat{Y}_{/W} \cong \overline{G}(\mathbb{Q})\backslash \bigsqcup_{\substack{\tau \in \overline{V} \\ \overline{Q}(\tau)=\alpha}} \left(\mathfrak{X}(\tau) \times \Omega(\tau)/\overline{U}^p\right).$$

The formal W-scheme $\mathfrak{M}$ in (7.2) is defined by replacing $\mathfrak{X}(\tau)$ by $\mathfrak{X}$ in the right hand side of (7.4), and $\mathfrak{M}^k$ is defined by replacing $\mathfrak{X}(\tau)$ by $\mathfrak{X}^k(\tau)$.

If Frob denotes the (arithmetic) Frobenius automorphism $W \to W$ then the formal schemes $\mathfrak{M}$ and $\mathfrak{M}^k$ come equipped with isomorphisms of formal W-schemes (described at the beginning of [12, §4.2])

$$\mathfrak{M}^{\mathrm{Frob}} \cong \mathfrak{M} \qquad \mathfrak{M}^{k,\mathrm{Frob}} \cong \mathfrak{M}^k,$$

which are compatible with the morphisms in (7.2), and with the evident isomorphisms

$$(\widehat{Y}_{/W})^{\mathrm{Frob}} \cong \widehat{Y}_{/W} \qquad (\widehat{M}_{/W})^{\mathrm{Frob}} \cong \widehat{M}_{/W}.$$

In other words, the entire diagram (7.2) is invariant under base change by Frob. By Grothendieck's theories of faithfully flat descent and formal GAGA, any coherent $\mathcal{O}_{\widehat{Y}_{/W}}$-module that is invariant under Frob descends to a coherent $\mathcal{O}_Y$-module. In particular if $\mathfrak{F}^1$ and $\mathfrak{F}^2$ are coherent sheaves on $\mathcal{O}_{\mathfrak{M}^1}$ and $\mathcal{O}_{\mathfrak{M}^2}$ respectively, each invariant under Frob, then for every ℓ the coherent $\mathcal{O}_{\mathfrak{M}}$-module $\mathrm{Tor}_\ell^{\mathcal{O}_{\mathfrak{M}}}(\mathfrak{F}^1, \mathfrak{F}^2)$ is invariant under Frob and is annihilated by the ideal sheaf of the closed formal subscheme

$$\widehat{Y}_{/W} \cong \mathfrak{M}^1 \times_{\mathfrak{M}} \mathfrak{M}^2 \hookrightarrow \mathfrak{M}.$$

Thus we may view $\mathrm{Tor}_\ell^{\mathcal{O}_{\mathfrak{M}}}(\mathfrak{F}^1, \mathfrak{F}^2)$ as a coherent $\mathcal{O}_Y$-module and form

(7.5) $$[\mathfrak{F}^1 \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathfrak{F}^2] = \sum_{\ell \geq 0}(-1)^\ell[\mathrm{Tor}_\ell^{\mathcal{O}_{\mathfrak{M}}}(\mathfrak{F}^1, \mathfrak{F}^2)] \in \mathbf{K}_0(Y).$$

For $k \in \{1, 2\}$ let $\mathfrak{B}^k$ be the largest ideal sheaf of $\mathcal{O}_{\mathfrak{M}^k}$ that is locally W-torsion, and define $\mathfrak{A}^k$ by the exactness of

$$0 \to \mathfrak{B}^k \to \mathcal{O}_{\mathfrak{M}^k} \to \mathfrak{A}^k \to 0.$$

We then have a decomposition in $\mathbf{K}_0(Y)$

(7.6) $$[\mathcal{O}_{\mathfrak{M}^1} \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathcal{O}_{\mathfrak{M}^2}] = [\mathfrak{O}_Y]^{\mathrm{hor}} + [\mathfrak{O}_Y]^{\mathrm{ver}}$$

in which

$$\begin{aligned}
[\mathfrak{O}_Y]^{\mathrm{hor}} &= [\mathfrak{A}^1 \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathfrak{A}^2] \\
[\mathfrak{O}_Y]^{\mathrm{ver}} &= [\mathfrak{A}^1 \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathfrak{B}^2] + [\mathfrak{B}^1 \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathfrak{A}^2] + [\mathfrak{B}^1 \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathfrak{B}^2].
\end{aligned}$$

By its construction, the class $[\mathfrak{O}_Y]^{\mathrm{ver}}$ may be viewed also as an element of $\mathbf{K}_0^{\mathrm{ver}}(Y)$.

Proposition 7.1. *There is a closed subscheme $J \to Y$ of dimension zero such that*

$$[\mathfrak{O}_Y]^{\mathrm{hor}} - [\mathcal{O}_Y]^\bullet - [\mathcal{O}_Y]^{\bullet\bullet} \in \mathrm{Image}\left(\mathbf{K}_0^J(Y) \to \mathbf{K}_0(Y)\right).$$

Proof. Given classes $[\mathcal{F}]$ and $[\mathcal{G}]$ in $\mathbf{K}_0(Y)$, write $[\mathcal{F}] \sim [\mathcal{G}]$ to mean that

$$[\mathcal{F}] - [\mathcal{G}] \in \text{Image}\big(\mathbf{K}_0^J(Y) \to \mathbf{K}_0(Y)\big)$$

for some closed subscheme $J \to Y$ of dimension zero. Let us say that a coherent $\mathcal{O}_Y$-module $\mathcal{A}$ is *essentially horizontal* if every point $\eta \in Y$ in the support of $\mathcal{A}$ is either the generic point of a horizontal component of Y or is a closed point of Y. That is, the support of an essentially horizontal $\mathcal{O}_Y$-module $\mathcal{A}$ contains no vertical components of dimension greater than zero. Note that any subquotient of an essentially horizontal $\mathcal{O}_Y$-module is again essentially horizontal. If $\mathcal{A}$ is essentially horizontal then an induction argument using the exact sequence of coherent $\mathcal{O}_Y$-modules

$$0 \to \mathcal{A}' \to \mathcal{A} \to \mathcal{A} \otimes_{\mathcal{O}_Y} \mathcal{O}_D \to 0$$

and the methods of [12, Lemma 2.2.1] show that, in the notation of §6,

$$(7.7) \qquad [\mathcal{A}] \sim \sum_{D \in \Pi^{\text{hor}}(Y)} \text{mult}_D(\mathcal{A}) \cdot [\mathcal{O}_D]$$

where $\Pi^{\text{hor}}(Y)$ is the set of irreducible horizontal components of Y, each endowed with its reduced subscheme structure.

Let $\mathcal{B}$ be the subsheaf of locally $\mathbb{Z}_p$-torsion sections of $\mathcal{O}_Y$ and define $\mathcal{A}$ by the exactness of

$$0 \to \mathcal{B} \to \mathcal{O}_Y \to \mathcal{A} \to 0.$$

The sheaf $\mathcal{A}$ has no $\mathbb{Z}_p$-torsion local sections, and the stalks of $\mathcal{A}$ are $\mathbb{Z}_p$-torsion free. To see that $\mathcal{A}$ is essentially horizontal, suppose that $\eta \in Y$ has residue characteristic p and Zariski closure of dimension greater than zero. This implies that no horizontal component of Y passes through η, and hence that every prime ideal of $\mathcal{O}_{Y,\eta}$ has residue characteristic p. We deduce that $\mathcal{O}_{Y,\eta}[1/p]$ is the trivial ring, and so $\mathcal{O}_{Y,\eta}$ is itself $\mathbb{Z}_p$-torsion. But then $\mathcal{A}_\eta$ is both $\mathbb{Z}_p$-torsion and $\mathbb{Z}_p$-torsion free, hence $\mathcal{A}_\eta = 0$ as desired. As the support of $\mathcal{B}$ is contained in the special fiber of Y, $\text{mult}_D(\mathcal{A}) = \text{mult}_D(\mathcal{O}_Y)$ for every $D \in \Pi^{\text{hor}}(Y)$, and we now deduce from (7.7) that

$$[\mathcal{A}] \sim [\mathcal{O}_Y]^\bullet + [\mathcal{O}_Y]^{\bullet\bullet}.$$

Fix a closed point $x \in \mathfrak{M}$, let R be the completion of the local ring $\mathcal{O}_{\mathfrak{M},x}$, let N^k be the completed stalk at x of the $\mathcal{O}_\mathfrak{M}$-module $\mathcal{O}_{\mathfrak{M}^k}$, and let P^k be the maximal W-torsion free quotient of N_k (which is isomorphic as an R-module to the completed stalk of $\mathfrak{A}^k$ at x). It follows from the proof of [12, Proposition 4.2.5] that

$$\text{Tor}_\ell^R(P^1, P^2) = 0$$

for all $\ell > 0$. Thus $\text{Tor}_\ell^{\mathcal{O}_\mathfrak{M}}(\mathfrak{A}^1, \mathfrak{A}^2)$ has trivial stalks, and so

$$[\mathfrak{D}_Y]^{\text{hor}} = [\mathfrak{A}^1 \otimes_{\mathcal{O}_\mathfrak{M}} \mathfrak{A}^2].$$

The proof of [12, Proposition 4.2.5] also shows that $P^1 \otimes_R P^2$ is free of finite rank over W; in other words the completed stalks at closed points of the $\mathcal{O}_{Y/W}$-module $\mathfrak{A}^1 \otimes_{\mathcal{O}_\mathfrak{M}} \mathfrak{A}^2$ are free of finite rank over W. It follows that all stalks of the $\mathcal{O}_{Y/W}$-module $\mathfrak{A}^1 \otimes_{\mathcal{O}_\mathfrak{M}} \mathfrak{A}^2$ are W-torsion free, from which one easily deduces from the faithful flatness of $\mathbb{Z}_p \to W$ that the local sections of the coherent $\mathcal{O}_Y$-module $\mathfrak{A}^1 \otimes_{\mathcal{O}_\mathfrak{M}} \mathfrak{A}^2$ are $\mathbb{Z}_p$-torsion free. The kernel of the evident surjection of $\mathcal{O}_Y$- modules

$$\mathcal{O}_{\mathfrak{M}^1} \otimes_{\mathcal{O}_\mathfrak{M}} \mathcal{O}_{\mathfrak{M}^2} \to \mathfrak{A}^1 \otimes_{\mathcal{O}_\mathfrak{M}} \mathfrak{A}^2$$

is generated by the ideal sheaves $\mathfrak{B}^1 \otimes_{\mathcal{O}_{\mathfrak{M}}} \mathcal{O}_{\mathfrak{M}^2}$ and $\mathcal{O}_{\mathfrak{M}^1} \otimes_{\mathcal{O}_{\mathfrak{M}}} \mathfrak{B}^2$, and so is locally $\mathbb{Z}_p$-torsion. Using the isomorphism $\mathcal{O}_Y \cong \mathcal{O}_{\mathfrak{M}^1} \otimes_{\mathcal{O}_{\mathfrak{M}}} \mathcal{O}_{\mathfrak{M}^2}$, we therefore deduce that $\mathfrak{A}^1 \otimes_{\mathcal{O}_{\mathfrak{M}}} \mathfrak{A}^2$ is the maximal quotient sheaf of $\mathcal{O}_Y$ whose local sections are $\mathbb{Z}_p$-torsion free. In other words $\mathfrak{A}^1 \otimes_{\mathcal{O}_{\mathfrak{M}}} \mathfrak{A}^2 \cong \mathcal{A}$. Thus $[\mathfrak{O}_Y]^{\mathrm{hor}} = [\mathcal{A}]$ and

$$[\mathfrak{O}_Y]^{\mathrm{hor}} - [\mathcal{O}_Y]^{\bullet} - [\mathcal{O}_Y]^{\bullet\bullet} \sim [\mathcal{A}] - [\mathcal{A}] = 0.$$

$\square$

Given a coherent $\mathcal{O}_Y$-module $\mathcal{F}$, we regard $\mathrm{Tor}_{\ell}^{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_{Y_0})$ as a coherent $\mathcal{O}_{Y_0}$-module and define

$$[\mathcal{F} \otimes_{\mathcal{O}_Y}^L \mathcal{O}_{Y_0}] = \sum_{\ell \geq 0} (-1)^{\ell} [\mathrm{Tor}_{\ell}^{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_{Y_0})] \in \mathbf{K}_0(Y_0).$$

This class depends only on the class $[\mathcal{F}]$, not on the sheaf $\mathcal{F}$ itself, and the construction $[\mathcal{F}] \mapsto [\mathcal{F} \otimes_{\mathcal{O}_Y}^L \mathcal{O}_{Y_0}]$ defines a homomorphism $\mathbf{K}_0(Y) \to \mathbf{K}_0(Y_0)$, as well as homomorphisms

$$\mathbf{K}_0^{\mathrm{ver}}(Y) \to \mathbf{K}_0^{\mathrm{ver}}(Y_0) \qquad \mathbf{K}_0(Y^{\bullet}) \to \mathbf{K}_0^{\mathrm{ver}}(Y_0).$$

From the decomposition (2.2) we deduce an isomorphism

$$\mathbf{K}_0^{\mathrm{ver}}(Y_0) \cong \bigoplus_{T \in \Sigma(\alpha)} \mathbf{K}_0^{\mathrm{ver}}(Z(T)).$$

Given a $T \in \Sigma(\alpha)$ and a coherent $\mathcal{O}_{Z(T)}$-module $\mathcal{F}_0$ that is annihilated by a power of p, define

$$\chi_T(\mathcal{F}_0) = \sum_{\ell \geq 0} (-1)^{\ell} \mathrm{length}_{\mathbb{Z}_p} R^{\ell} \mu_* \mathcal{F}_0$$

where $\mu : Z(T) \to \mathrm{Spec}(\mathbb{Z}_p)$ is the structure morphism. If $\mathcal{F}_0$ is supported in dimension zero then it is easy to see that

$$\chi_T(\mathcal{F}_0) = \sum_{y \in Z(T)(\mathbb{F}_p^{\mathrm{alg}})} \mathrm{length}_{\mathcal{O}_{Y,y}} \mathcal{F}_{0,y}.$$

As χ_T depends only on the class $[\mathcal{F}_0] \in \mathbf{K}_0^{\mathrm{ver}}(Z(T))$, we may extend χ_T to a homomorphism $\mathbf{K}_0^{\mathrm{ver}}(Z(T)) \to \mathbb{Z}$ and define, for any class $[\mathcal{F}]$ in either $\mathbf{K}_0^{\mathrm{ver}}(Y)$ or $\mathbf{K}_0(Y^{\bullet})$, the intersection multiplicity of $\mathcal{F}$ and $\mathcal{O}_{M_0}$ at $Z(T)$ by

$$I_{\mathcal{O}_{Z(T)}}(\mathcal{F}, \mathcal{O}_{M_0}) = \chi_T(\mathcal{F} \otimes_{\mathcal{O}_Y}^L \mathcal{O}_{Y_0}).$$

As in [12, §4.3], define formal W-schemes for $k \in \{1, 2\}$

$$\mathfrak{M}_0 = \widehat{M}_{0/W} \times_{\widehat{M}_{/W}} \mathfrak{M} \qquad \mathfrak{M}_0^k = \widehat{M}_{0/W} \times_{\widehat{M}_{/W}} \mathfrak{M}^k$$

so that

$$\widehat{Y}_{0/W} \cong \mathfrak{M}_0 \times_{\mathfrak{M}} \widehat{Y}_{/W} \cong \mathfrak{M}_0^1 \times_{\mathfrak{M}_0} \mathfrak{M}_0^2.$$

These formal schemes admit Čerednik-Drinfeld style uniformizations: if we set $\Omega_0(\tau) = \Omega(\tau) \cap \overline{G}_0(\mathbb{A}_f^p)$ then

$$(7.8) \qquad \mathfrak{M}_0 \cong \overline{G}_0(\mathbb{Q}) \backslash \bigsqcup_{\substack{\tau \in \overline{V} \\ \overline{Q}(\tau) = \alpha}} \left(\mathfrak{X}_0 \times \Omega_0(\tau) \overline{U}^{\mathrm{max},p} / \overline{U}^p \right)$$

where the product $\Omega_0(\tau)\overline{U}^{\mathrm{max},p}$ is taken inside of $\overline{G}(\mathbb{A}_f^p)$. Similarly

(7.9) $$\mathfrak{M}_0^k \cong \overline{G}_0(\mathbb{Q})\backslash \bigsqcup_{\substack{\tau \in \overline{V} \\ \overline{Q}(\tau)=\alpha}} \left(\mathfrak{X}_0(\tau_k) \times \Omega_0(\tau)\overline{U}^{\mathrm{max},p}/\overline{U}^p \right).$$

Any coherent $\mathcal{O}_Y$-module $\mathcal{F}$ may be viewed as a Frob-invariant coherent $\mathcal{O}_{\mathfrak{M}}$-module annihilated by the ideal sheaf of the closed formal subscheme $\widehat{Y}_{/W} \to \mathfrak{M}$. For every $\ell \geq 0$ we may then form $\mathrm{Tor}_\ell^{\mathcal{O}_{\mathfrak{M}}}(\mathcal{F}, \mathcal{O}_{\mathfrak{M}_0})$, a Frob-invariant coherent $\mathcal{O}_{\mathfrak{M}_0}$-module annihilated by the ideal sheaf of the closed formal subscheme $\widehat{Y}_{0/W} \to \mathcal{O}_{\mathfrak{M}_0}$. As before, using formal GAGA and faithfully flat descent we view $\mathrm{Tor}_\ell^{\mathcal{O}_{\mathfrak{M}}}(\mathcal{F}, \mathcal{O}_{\mathfrak{M}_0})$ as a coherent $\mathcal{O}_Y$-module, and define

$$[\mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathcal{O}_{\mathfrak{M}_0}] = \sum_{\ell \geq 0}(-1)^\ell [\mathrm{Tor}_\ell^{\mathcal{O}_{\mathfrak{M}}}(\mathcal{F}, \mathcal{O}_{\mathfrak{M}_0})] \in \mathbf{K}_0(Y_0).$$

It is easy to check that $[\mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathcal{O}_{\mathfrak{M}_0}] = [\mathcal{F} \otimes_{\mathcal{O}_Y}^L \mathcal{O}_{Y_0}]$.

For any $T \in \mathrm{Sym}_2(\mathbb{Z})^\vee$ let $\overline{V}_0(T)$ be the set of pairs $(s_1, s_2) \in \overline{V}_0 \times \overline{V}_0$ for which (2.1) holds, where $[s_i, s_j] = -\mathrm{Tr}(s_i s_j)$ is the bilinear form on $\overline{V}_0$ satisfying $[s, s] = 2\overline{Q}_0(s)$. Given a pair $(s_1, s_2) \in \overline{V}_0(T)$ set

(7.10) $$\tau = s_1 \varpi_1 + s_2 \varpi_2 \in \overline{V}_0 \otimes_{\mathbb{Q}} F \cong \overline{V}$$

and let (τ_1, τ_2) be the image of τ under (7.3). Assuming that $\det(T) \neq 0$, Kudla-Rapoport [18] (and Kudla-Rapoport-Yang [19] when $p = 2$) compute the intersection multiplicity of the divisors $\mathfrak{h}_m(\tau_1)$ and $\mathfrak{h}_m(\tau_2)$ in the Drinfeld space $\mathfrak{h}_m$. This intersection multiplicity depends only the isomorphism class of the rank two quadratic space

$$\mathbb{Z}_p\tau_1 + \mathbb{Z}_p\tau_2 = \mathbb{Z}_p s_1 + \mathbb{Z}_p s_2 \subset \overline{V}_0 \otimes_{\mathbb{Q}} \mathbb{Q}_p,$$

which is determined by T. Let $e_p(T)$ be this intersection multiplicity, as in [18, Theorem 6.1]. In the notation of [20, Chapter 7.6], $\nu_p(T) = 2e_p(T)$.

Proposition 7.2. *Define a $\mathbb{Z}[1/p]$-lattice $\overline{L}_0 \subset \overline{V}_0$ by*

$$\overline{L}_0 = \{v \in \overline{V}_0 : \widehat{\Lambda}_0^p \cdot \iota_0(v) \subset \widehat{\Lambda}_0^p\}$$

and a discrete subgroup $\overline{\Gamma}_0 \subset \overline{G}_0(\mathbb{Q})$ by

$$\overline{\Gamma}_0 = \{\gamma \in \overline{G}_0(\mathbb{Q}) : \widehat{\Lambda}_0^p \cdot \iota_0(g) \subset \widehat{\Lambda}_0^p\}.$$

Then for every nonsingular $T \in \Sigma(\alpha)$

$$I_{\mathcal{O}_{Z(T)}}([\mathcal{O}_Y]^\bullet, \mathcal{O}_{M_0}) + I_{\mathcal{O}_{Z(T)}}([\mathfrak{D}_Y]^{\mathrm{ver}}, \mathcal{O}_{M_0}) = |H| \cdot e_p(T) \cdot |\overline{\Gamma}_0 \backslash \overline{L}_0(T)|$$

where $L_0(T) = \{(s_1, s_2) \in V_0(T) : s_1, s_2 \in L_0\}$ and $\overline{\Gamma}_0$ acts on $L_0(T)$ by conjugation.

Proof. As T is nonsingular, the scheme $Z(T)$ is supported in characteristic p by [20, Theorem 3.6.1]. Thus

$$\mathbf{K}_0^{\mathrm{ver}}(Z(T)) = \mathbf{K}_0(Z(T)).$$

and $I_{\mathcal{O}_{Z(T)}}(\mathcal{F}, \mathcal{O}_{M_0})$ is defined for every class $[\mathcal{F}] \in \mathbf{K}_0(Y)$. By the decomposition (6.1) each $D \in \Pi^{\bullet\bullet}(Y)$ is contained in $Z(T')$ for some $T' \in \Sigma(\alpha)$. In particular

$Z(T')$ has nonempty generic fiber, and hence $T' \neq T$. Thus $D \cap Z(T) = \emptyset$ and the coherent $\mathcal{O}_{Y_0}$-module $\operatorname{Tor}_\ell^{\mathcal{O}_Y}(\mathcal{O}_D, \mathcal{O}_{Y_0})$ has trivial restriction to $Z(T)$. Thus

$$(7.11) \quad I_{\mathcal{O}_{Z(T)}}([\mathcal{O}_Y]^{\bullet\bullet}, \mathcal{O}_{M_0}) = \sum_{D \in \Pi^{\bullet\bullet}(Y)} \operatorname{mult}_D(\mathcal{O}_Y) \cdot \chi_T(\operatorname{Tor}_\ell^{\mathcal{O}_Y}(\mathcal{O}_D, \mathcal{O}_{Y_0})) = 0.$$

If $[\mathcal{F}]$ lies in the image of $\mathbf{K}_0^J(Y) \to \mathbf{K}_0(Y)$ for $J \to Y$ a closed subscheme of dimension zero, then $I_{\mathcal{O}_{Z(T)}}([\mathcal{F}], \mathcal{O}_{M_0}) = 0$ by [12, Lemma 4.2.2]; hence combining (7.11) with Proposition 7.1 and (7.6) yields

$$
\begin{aligned}
& I_{\mathcal{O}_{Z(T)}}([\mathcal{O}_Y]^\bullet, \mathcal{O}_{M_0}) + I_{\mathcal{O}_{Z(T)}}([\mathfrak{D}_Y]^{\operatorname{ver}}, \mathcal{O}_{M_0}) \\
&= \quad I_{\mathcal{O}_{Z(T)}}([\mathfrak{D}_Y]^{\operatorname{hor}}, \mathcal{O}_{M_0}) + I_{\mathcal{O}_{Z(T)}}([\mathfrak{D}_Y]^{\operatorname{ver}}, \mathcal{O}_{M_0}) \\
&= \quad I_{\mathcal{O}_{Z(T)}}(\mathcal{O}_{\mathfrak{M}^1} \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathcal{O}_{\mathfrak{M}^2}, \mathcal{O}_{M_0}) \\
&= \quad \chi_T\big((\mathcal{O}_{\mathfrak{M}^1} \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathcal{O}_{\mathfrak{M}^2}) \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathcal{O}_{\mathfrak{M}_0}\big).
\end{aligned}
$$

The equality

$$\chi_T\big((\mathcal{O}_{\mathfrak{M}^1} \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathcal{O}_{\mathfrak{M}^2}) \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathcal{O}_{\mathfrak{M}_0}\big) = |H| \cdot e_p(T) \cdot |\overline{\Gamma}_0 \backslash \overline{L}_0(T)|$$

is now proved exactly as in [12, §4.3]; see especially Lemma 4.3.2 and Proposition 4.3.4 of [loc. cit.].   □

Suppose $T \in \Sigma(\alpha)$ is singular and denote by t_1 and t_2 the diagonal entries of T. Let n_1, n_2, and t be as in (6.7) so that $\alpha = (n_1 \varpi_1 + n_2 \varpi_2)^2 \cdot t$ and $K = \mathbb{Q}(\sqrt{-t})$ is a quadratic imaginary subfield of $F(\sqrt{-\alpha})$. Abbreviate $d_K = \operatorname{disc}(K/\mathbb{Q})$ and define $n \in \mathbb{Z}^+$ by $4t = -n^2 d_K$. As in (6.8) there is an isomorphism of stacks $\mathcal{Z}(t) \cong \mathcal{Z}(T)$, which takes the triple $(\mathbf{A}_0, \lambda_0, s_0)$ to the quadruple $(\mathbf{A}_0, \lambda_0, n_1 s_0, n_2 s_0)$.

Proposition 7.3. *For every singular $T \in \Sigma(\alpha)$*

$$(7.12) \qquad I_{\mathcal{O}_{Z(T)}}([\mathcal{O}_Y]^\bullet, \mathcal{O}_{M_0}) = \frac{1}{2} \cdot |Z(T)(\mathbb{Q}_p^{\operatorname{alg}})| \cdot \operatorname{ord}_p(d_K).$$

Proof. For each $D \in \Pi^\bullet(Y)$ the scheme $D \times_Y Y_0$ has dimension zero. It follows that the coherent $\mathcal{O}_{Y_0}$-module $\operatorname{Tor}_\ell^{\mathcal{O}_Y}(\mathcal{O}_D, \mathcal{O}_{Y_0})$ is supported in dimension zero, and the left hand side of (7.12) is equal to

$$(7.13) \qquad \sum_{y \in Z(T)(\mathbb{F}_p^{\operatorname{alg}})} \sum_{D \in \Pi^\bullet(Y)} \operatorname{mult}_D(\mathcal{O}_Y) \cdot \sum_{\ell \geq 0} \operatorname{length}_{\mathcal{O}_{Y,y}} \operatorname{Tor}_\ell^{\mathcal{O}_{Y,y}}(\mathcal{O}_{D,y}, \mathcal{O}_{Y_0,y}).$$

Given a $y \in Z(T)(\mathbb{F}_p^{\operatorname{alg}})$ let R_M be the completion of the strictly Henselian local ring of M at y, and define R_Y, R_{M_0}, and R_{Y_0} similarly. Let $\Pi^\bullet(R_Y)$ be the set of minimal primes R_Y of residue characteristic 0 that do not come from R_{Y_0} (more precisely: do not contain the kernel of the surjection $R_Y \to R_{Y_0}$). As in (6.10) the left hand side of (7.13) is equal to

$$(7.14) \qquad \sum_{y \in Z(T)(\mathbb{F}_p^{\operatorname{alg}})} \sum_{\mathfrak{p} \in \Pi^\bullet(R_Y)} \operatorname{mult}(\mathfrak{p}) \cdot \operatorname{length}_{R_Y}(R_Y/\mathfrak{p} \otimes_{R_M} R_{M_0}).$$

From the uniformizations (7.1) and (7.4) and similar uniformizations (as in [12, §4]) of $\widehat{M}_{0/W}$ and $\widehat{Y}_{0/W}$, we deduce that there are $m \in \mathbb{Z}$, $x \in \mathfrak{h}_m(\mathbb{F}_p^{\operatorname{alg}})$, and $s_0 \in \overline{V}_0$

with $\overline{Q}_0(s_0) = t$ for which

$$
\begin{aligned}
R_{M_0} &\cong \widehat{\mathcal{O}}_{\mathfrak{h}_m,x} \\
R_M &\cong \widehat{\mathcal{O}}_{\mathfrak{h}_m,x} \widehat{\otimes}_W \widehat{\mathcal{O}}_{\mathfrak{h}_m,x} \\
R_{Y_0} &\cong \widehat{\mathcal{O}}_{\mathfrak{h}_m(s_0),x} \\
R_Y &\cong \widehat{\mathcal{O}}_{\mathfrak{h}_m(\tau_1),x} \widehat{\otimes}_W \widehat{\mathcal{O}}_{\mathfrak{h}_m(\tau_2),x}
\end{aligned}
$$

where $\tau = (n_1\varpi_1 + n_2\varpi_2)s_0$ and, as always, (τ_1, τ_2) is the image of τ under (7.3). Letting (n_1, n_2) be the image of $(n_1\varpi_1 + n_2\varpi_2)$ under $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathbb{Z}_p \times \mathbb{Z}_p$, we see that $\tau_i = n_i s_0$, and from the fact that $\gcd(n_1, n_2) = 1$ we deduce that at least one of n_1, n_2 lies in $\mathbb{Z}_p^\times$. Hence there are natural surjections

$$
\widehat{\mathcal{O}}_{\mathfrak{h}_m(\tau_1),x} \to \widehat{\mathcal{O}}_{\mathfrak{h}_m(s_0),x} \qquad \widehat{\mathcal{O}}_{\mathfrak{h}_m(\tau_2),x} \to \widehat{\mathcal{O}}_{\mathfrak{h}_m(s_0),x}
$$

at least one of which is an isomorphism. The completed local rings $\widehat{\mathcal{O}}_{\mathfrak{h}_m,x}$ and $\widehat{\mathcal{O}}_{\mathfrak{h}_m(s_0),x}$ are described in detail in [18] (at least for $p \neq 2$; for $p = 2$ the calculations are in [19]). In the notation of [18], the point x may be either *ordinary* or *superspecial*. From Propositions 3.2 and 3.3 of [18] (and the appendix to [19, §11] for $p = 2$), we see that there are three mutually exclusive possibilities:

(a) $\widehat{\mathcal{O}}_{\mathfrak{h}_m(s_0),x}$ is W-torsion;
(b) x is ordinary, p is inert in K, and the quotient of the W-algebra $\widehat{\mathcal{O}}_{\mathfrak{h}_m(s_0),x}$ by its ideal of W-torsion is isomorphic to W;
(c) x is supersingular, p is ramified in K, and the quotient of the W-algebra $\widehat{\mathcal{O}}_{\mathfrak{h}_m(s_0),x}$ by its ideal of W-torsion is isomorphic to $\mathcal{W}$, where $\mathcal{W}$ is the ring of integers in $K \otimes_{\mathbb{Q}} \mathrm{Frac}(W)$.

The same statements hold verbatim with s_0 replaced by τ_1 or by τ_2.

Suppose first that p is unramified in K and choose a $y \in Z(T)(\mathbb{F}_p^{\mathrm{alg}})$. From the above it follows that either R_{Y_0} and R_Y are both W-torsion, or the quotients of R_{Y_0} and R_Y by their ideals of W-torsion are both isomorphic to W. In the latter case each of R_{Y_0} and R_Y has a unique prime ideal of characteristic 0. In either case every prime ideal of R_Y of residue characteristic 0 comes from R_{Y_0}, and so $\Pi^\bullet(R_Y) = \emptyset$. We deduce that if p is unramified in K then $\Pi^\bullet(R_Y) = \emptyset$ for every $y \in Z(T)(\mathbb{F}_p^{\mathrm{alg}})$, and hence the left hand side of (7.14) is 0. From this we see that both sides of (7.12) are zero, and we are done.

Now suppose that p is ramified in K and again choose a $y \in Z(T)(\mathbb{F}_p^{\mathrm{alg}})$. Then either R_{Y_0} and R_Y are both W-torsion, or the quotient of R_{Y_0} by its ideal of W-torsion is isomorphic to $\mathcal{W}$ and the quotient of R_Y by its ideal of W-torsion is isomorphic to $\mathcal{W} \otimes_W \mathcal{W}$. Assume we are in the latter case. Then R_Y has exactly two prime ideals of residue characteristic 0, call them $\mathfrak{p}$ and $\mathfrak{q}$. The prime $\mathfrak{p}$ is the kernel of the surjection

$$
R_Y \to \mathcal{W} \otimes_W \mathcal{W} \xrightarrow{a \otimes b \mapsto ab} \mathcal{W}
$$

while the prime $\mathfrak{q}$ is the kernel of the surjection

$$
R_Y \to \mathcal{W} \otimes_W \mathcal{W} \xrightarrow{a \otimes b \mapsto a\overline{b}} \mathcal{W}
$$

in which $b \mapsto \overline{b}$ is the nontrivial W-algebra automorphism of $\mathcal{W}$. Note that the quotient map $R_Y \to R_Y/\mathfrak{p}$ factors through the surjection $R_Y \to R_{Y_0}$, but that

$R_Y \to R_Y/\mathfrak{q}$ does not. In other words $\mathfrak{p}$ comes from R_{Y_0} while $\mathfrak{q}$ does not, and hence $\Pi^\bullet(R_Y) = \{\mathfrak{q}\}$. Furthermore

$$R_Y/\mathfrak{q} \otimes_{R_M} R_{M_0} \cong R_Y/\mathfrak{q} \otimes_{(R_{M_0}\widehat{\otimes}R_{M_0})} R_{M_0} \cong \mathcal{W} \otimes_{(\mathcal{W}\otimes_W \mathcal{W})} \mathcal{W}$$

where in the final tensor product $\mathcal{W}$ is regarded as a $\mathcal{W} \otimes_W \mathcal{W}$ module in two distinct ways: on the left through $a \otimes b \mapsto a\overline{b}$ and on the right through $a \otimes b \mapsto ab$. From standard properties of the discriminant (for example [24, p. 64]), we deduce

$$\begin{aligned}
\text{length}_{R_{Y_0}}(R_Y/\mathfrak{q} \otimes_{R_M} R_{M_0}) &= \text{length}_{\mathcal{W}}(\mathcal{W} \otimes_{(\mathcal{W}\otimes_W \mathcal{W})} \mathcal{W}) \\
&= v_W(\text{disc}(\mathcal{W}/W)) \\
&= \text{ord}_p(d_K)
\end{aligned}$$

where v_W is the normalized valuation on W. It is easy to see that the localization of R_Y at $\mathfrak{q}$ is isomorphic to the fraction field of $\mathcal{W}$, and hence $\text{mult}(\mathfrak{q}) = 1$. Thus, in the case of p ramified in K, for every $y \in Z(T)(\mathbb{F}_p^{\text{alg}})$ either R_{Y_0} and R_Y are both W-torsion or

$$\sum_{\mathfrak{p}\in\Pi^\bullet(R_Y)} \text{mult}(\mathfrak{p}) \cdot \text{length}_{R_Y}(R_Y/\mathfrak{p} \otimes_{R_M} R_{M_0}) = \text{ord}_p(d_K).$$

Still assuming that p is ramified in K, we must count the number of $y \in Z(T)(\mathbb{F}_p^{\text{alg}})$ for which R_{Y_0} contains a prime ideal of residue characteristic 0. By the discussion above, when such a prime ideal exists it is unique and has residue field a degree two extension of the fraction field of W. Thus each such y has two distinct lifts to $Y_0(\mathbb{Q}_p^{\text{alg}})$, each of which must be contained in $Z(T)(\mathbb{Q}_p^{\text{alg}})$ by the decomposition (6.1). The number of y for which R_{Y_0} contains a prime ideal of residue characteristic 0 is therefore $\frac{1}{2}|Z(T)(\mathbb{Q}_p^{\text{alg}})|$. It follows that (7.14) is equal to $\frac{1}{2}|Z(T)(\mathbb{Q}_p^{\text{alg}})|\text{ord}_p(d_K)$, and (7.12) follows. $\qquad\square$

Given a coherent $\mathcal{O}_{\mathfrak{M}_0^k}$-module $\mathfrak{F}_0^k$ for $k \in \{1,2\}$ the sheaf $\text{Tor}_\ell^{\mathcal{O}_{\mathfrak{M}_0}}(\mathfrak{F}_0^1, \mathfrak{F}_0^2)$ is annihilated by the ideal sheaf of the closed formal subscheme $\widehat{Y}_{0/W} \to \mathfrak{M}_0$, and by formal GAGA may be viewed as a coherent $\mathcal{O}_{Y_{0/W}}$-module. If the sheaves $\mathfrak{F}_0^1$ and $\mathfrak{F}_0^2$ are each invariant under Frob then exactly as in (7.5) we may form

$$(7.15) \qquad [\mathfrak{F}_0^1 \otimes_{\mathcal{O}_{\mathfrak{M}_0}}^L \mathfrak{F}_0^2] = \sum_{\ell\geq 0}(-1)^\ell[\text{Tor}_\ell^{\mathcal{O}_{\mathfrak{M}_0}}(\mathfrak{F}_0^1, \mathfrak{F}_0^2)] \in \mathbf{K}_0(Y_0).$$

If either of $\mathfrak{F}_0^1$ or $\mathfrak{F}_0^2$ is locally W-torsion then (7.15) also defines a class in $\mathbf{K}_0^{\text{ver}}(Y_0)$.

Lemma 7.4. *Let $\mathfrak{B}_0^k$ be the ideal sheaf of locally W-torsion sections of $\mathcal{O}_{\mathfrak{M}_0^k}$ and define $\mathfrak{A}_0^k$ by the exactness of*

$$0 \to \mathfrak{B}_0^k \to \mathcal{O}_{\mathfrak{M}_0^k} \to \mathfrak{A}_0^k \to 0.$$

Then for any $T \in \Sigma(\alpha)$

$$\begin{aligned}
&I_{\mathcal{O}_{Z(T)}}([\mathfrak{D}_Y]^{\text{ver}}, \mathcal{O}_{M_0}) \\
&= \chi_T(\mathfrak{B}_0^1 \otimes_{\mathcal{O}_{\mathfrak{M}_0}}^L \mathfrak{B}_0^2) + \chi_T(\mathfrak{B}_0^1 \otimes_{\mathcal{O}_{\mathfrak{M}_0}}^L \mathfrak{A}_0^2) + \chi_T(\mathfrak{A}_0^1 \otimes_{\mathcal{O}_{\mathfrak{M}_0}}^L \mathfrak{B}_0^2).
\end{aligned}$$

Proof. From the definitions we have

$$\begin{aligned}
&I_{\mathcal{O}_{Z(T)}}([\mathfrak{D}_Y]^{\text{ver}}, \mathcal{O}_{M_0}) = \chi_T\big((\mathfrak{B}^1 \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathfrak{B}^2) \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathcal{O}_{\mathfrak{M}_0}\big) \\
&\qquad + \chi_T\big((\mathfrak{B}^1 \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathfrak{A}^2) \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathcal{O}_{\mathfrak{M}_0}\big) + \chi_T\big((\mathfrak{A}^1 \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathfrak{B}^2) \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathcal{O}_{\mathfrak{M}_0}\big).
\end{aligned}$$

The same argument used in the proof of [12, Lemma 4.3.2] shows that

$$\mathrm{Tor}_i^{\mathcal{O}_{\mathfrak{M}}}(\mathfrak{B}^k, \mathcal{O}_{\mathfrak{M}_0}) \cong \begin{cases} \mathfrak{B}_0^k & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

and hence

$$\begin{aligned} \chi_T\big((\mathfrak{B}^1 \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathfrak{B}^2) \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathcal{O}_{\mathfrak{M}_0}\big) &= \chi_T\big((\mathfrak{B}^1 \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathcal{O}_{\mathfrak{M}_0}) \otimes_{\mathcal{O}_{\mathfrak{M}_0}}^L (\mathfrak{B}^2 \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathcal{O}_{\mathfrak{M}_0})\big) \\ &= \chi_T(\mathfrak{B}_0^1 \otimes_{\mathcal{O}_{\mathfrak{M}_0}}^L \mathfrak{B}_0^2). \end{aligned}$$

Similarly

$$\mathrm{Tor}_i^{\mathcal{O}_{\mathfrak{M}}}(\mathfrak{A}^k, \mathcal{O}_{\mathfrak{M}_0}) \cong \begin{cases} \mathfrak{A}_0^k & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

implies that

$$\chi_T\big((\mathfrak{B}^1 \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathfrak{A}^2) \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathcal{O}_{\mathfrak{M}_0}\big) = \chi_T(\mathfrak{B}_0^1 \otimes_{\mathcal{O}_{\mathfrak{M}_0}}^L \mathfrak{A}_0^2)$$

and

$$\chi_T\big((\mathfrak{A}^1 \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathfrak{B}^2) \otimes_{\mathcal{O}_{\mathfrak{M}}}^L \mathcal{O}_{\mathfrak{M}_0}\big) = \chi_T(\mathfrak{A}_0^1 \otimes_{\mathcal{O}_{\mathfrak{M}_0}}^L \mathfrak{B}_0^2)$$

proving the claim. $\qquad\square$

The equality of the following proposition is derived from calculations of Kudla-Rapoport-Yang [19, 20].

Proposition 7.5. *If* $T \in \Sigma(\alpha)$ *is singular then*

$$\frac{1}{|H|} I_{\mathcal{O}_{Z(T)}}([\mathfrak{D}_Y]^{\mathrm{ver}}, \mathcal{O}_{M_0}) + \frac{h_{\widehat{\omega}_0}(\mathcal{Z}^{\mathrm{ver}}(t)_p)}{\log(p)} = \frac{1}{2} \deg_{\mathbb{Q}}(\mathcal{Z}(t)) \cdot \mathrm{ord}_p\left(\frac{4\alpha\alpha^\sigma}{td_K}\right).$$

Proof. Using the uniformizations (7.8) and (7.9) and the isomorphism

$$\widehat{Y}_{0/W} \cong \mathfrak{M}_0^1 \times_{\mathfrak{M}_0} \mathfrak{M}_0^2$$

we find

$$\widehat{Y}_{0/W} \cong \overline{G}_0(\mathbb{Q}) \backslash \bigsqcup_{\substack{\tau \in \overline{V} \\ \overline{Q}(\tau) = \alpha}} \left(\mathfrak{X}_0(\tau) \times \Omega_0(\tau) \overline{U}^{\mathrm{max},p} / \overline{U}^p\right)$$

where $\mathfrak{X}_0(\tau) = \mathfrak{X}_0(\tau_1) \times_{\mathfrak{X}_0} \mathfrak{X}_0(\tau_2)$ is the locus in $\mathfrak{X}$ where both of the quasi-endomorphisms τ_1 and τ_2 are integral. Using the notation of (7.10) there is a bijection

$$\{\tau \in \overline{V} : \overline{Q}(\tau) = \alpha\} \to \bigsqcup_{T \in \Sigma(\alpha)} \overline{V}_0(T)$$

given by $\tau \mapsto (s_1, s_2)$. Using the decomposition (6.1) to view $Z(T)$ as a closed subscheme of Y_0 for each $T \in \Sigma(\alpha)$, we identify $\widehat{Z}(T)_{/W}$ with the open and closed formal subscheme of $\widehat{Y}_{0/W}$

$$\widehat{Z}(T)_{/W} \cong \overline{G}_0(\mathbb{Q}) \backslash \bigsqcup_{(s_1, s_2) \in \overline{V}_0(T)} \left(\mathfrak{X}_0(\tau) \times \Omega_0(\tau) \overline{U}^{\mathrm{max},p} / \overline{U}^p\right).$$

Now fix a singular $T \in \Sigma(\alpha)$. Using the isomorphism $Z(T) \cong Z(t)$, we find

$$\widehat{Z}(T)_{/W} \cong \overline{G}_0(\mathbb{Q}) \backslash \bigsqcup_{\substack{s_0 \in \overline{V}_0 \\ \overline{Q}_0(s_0) = t}} \left(\mathfrak{X}_0(\tau) \times \Omega_0(\tau) \overline{U}^{\mathrm{max},p} / \overline{U}^p\right)$$

in which $\tau = (n_1\varpi_1 + n_2\varpi_2)s_0$. By noting that the factor $n_1\varpi_1 + n_2\varpi_2 \in \mathcal{O}_F$ is not divisible by any rational prime (as $\gcd(n_1, n_2) = 1$), we deduce that

$$
\begin{aligned}
\Omega_0(\tau) &= \{g \in \overline{G}_0(\mathbb{A}_f^p) : \widehat{\Lambda}^p \cdot \iota(g^{-1}\tau g) \in \widehat{\Lambda}^p\} \\
&= \{g \in \overline{G}_0(\mathbb{A}_f^p) : \iota(g^{-1}\tau g) \in \widehat{\mathcal{O}}_B^p\} \\
&= \{g \in \overline{G}_0(\mathbb{A}_f^p) : \iota(g^{-1}s_0 g) \in \widehat{\mathcal{O}}_{B_0}^p\}.
\end{aligned}
$$

In particular $\Omega_0(\tau) = \Omega_0(s_0)$. We now argue as in [19, §11]. By the Noether-Skolem theorem the action of $\overline{G}_0(\mathbb{Q})$ on $\{s_0 \in \overline{V}_0 : \overline{Q}_0(s_0) = t\}$ is transitive. Fixing one $s_0 \in \overline{V}_0$ with $\overline{Q}_0(s_0) = t$, embed $K \to \overline{B}_0$ via $\sqrt{-t} \mapsto s_0$. This induces a homomorphism $K^\times \to \overline{G}_0(\mathbb{Q})$ and, using the fact that $K^\times$ is the stabilizer of s_0 in $\overline{G}_0(\mathbb{Q})$, we deduce

$$(7.16) \qquad \widehat{Z}(T)_{/W} \cong K^\times \backslash \big(\mathfrak{X}_0(\tau) \times \Omega_0(s_0)\overline{U}^{\mathrm{max},p}/\overline{U}^p\big).$$

Assume that p is inert in K, set

$$K^\flat = \{x \in K^\times : \mathrm{ord}_p(\mathrm{N}_{K/\mathbb{Q}}(x)) = 0\},$$

and write

$$\mathfrak{h}_m(\tau) = \mathfrak{h}_m(\tau_1) \times_{\mathfrak{h}_m} \mathfrak{h}_m(\tau_2).$$

As in [19, (11.8)] rewrite (7.16) as

$$(7.17) \qquad \widehat{Z}(T)_{/W} \cong \big(\mathfrak{h}_0(\tau) \sqcup \mathfrak{h}_1(\tau)\big) \times \big(K^\flat \backslash \Omega_0(s_0)\overline{U}^{\mathrm{max},p}/\overline{U}^p\big).$$

For $k \in \{1, 2\}$ define the coherent $\mathcal{O}_{\mathfrak{h}_m}$-module $\mathcal{O}_{\mathfrak{h}_m(\tau_k)}^{\mathrm{ver}}$ to be the ideal sheaf of locally W-torsion sections of the sheaf $\mathcal{O}_{\mathfrak{h}_m(\tau_k)}$, and define $\mathcal{O}_{\mathfrak{h}_m(\tau_k)}^{\mathrm{hor}}$ by the exactness of

$$0 \to \mathcal{O}_{\mathfrak{h}_m(\tau_k)}^{\mathrm{ver}} \to \mathcal{O}_{\mathfrak{h}_m(\tau_k)} \to \mathcal{O}_{\mathfrak{h}_m(\tau_k)}^{\mathrm{hor}} \to 0.$$

Under the uniformization (7.17) we have isomorphisms of coherent $\mathcal{O}_{\widehat{Z}(T)_{/W}}$-modules

$$
\begin{aligned}
\mathrm{Tor}_\ell^{\mathcal{O}_{\mathfrak{M}_0}}(\mathfrak{B}_0^1, \mathfrak{B}_0^2) &\cong \mathrm{Tor}_\ell^{\mathcal{O}_{\mathfrak{h}_m}}(\mathcal{O}_{\mathfrak{h}_m(\tau_1)}^{\mathrm{ver}}, \mathcal{O}_{\mathfrak{h}_m(\tau_2)}^{\mathrm{ver}}) \\
\mathrm{Tor}_\ell^{\mathcal{O}_{\mathfrak{M}_0}}(\mathfrak{B}_0^1, \mathfrak{A}_0^2) &\cong \mathrm{Tor}_\ell^{\mathcal{O}_{\mathfrak{h}_m}}(\mathcal{O}_{\mathfrak{h}_m(\tau_1)}^{\mathrm{ver}}, \mathcal{O}_{\mathfrak{h}_m(\tau_2)}^{\mathrm{hor}}) \\
\mathrm{Tor}_\ell^{\mathcal{O}_{\mathfrak{M}_0}}(\mathfrak{A}_0^1, \mathfrak{B}_0^2) &\cong \mathrm{Tor}_\ell^{\mathcal{O}_{\mathfrak{h}_m}}(\mathcal{O}_{\mathfrak{h}_m(\tau_1)}^{\mathrm{hor}}, \mathcal{O}_{\mathfrak{h}_m(\tau_2)}^{\mathrm{ver}}).
\end{aligned}
$$

Letting $\mu : \mathfrak{h}_m \to \mathrm{Spf}(W)$ denote the structure map and

$$\chi(\mathfrak{F}) = \sum_{k \geq 0} \mathrm{length}_W R^k \mu_* \mathfrak{F}$$

the Euler characteristic of a coherent, properly supported, locally W-torsion $\mathcal{O}_{\mathfrak{h}(m)}$-module $\mathfrak{F}$, Kudla-Rapoport-Yang have proved (see the proof of [20, Proposition 7.6.4])

$$
\begin{aligned}
\sum_{\ell \geq 0} \chi\big(\mathrm{Tor}_\ell^{\mathcal{O}_{\mathfrak{h}_m}}(\mathcal{O}_{\mathfrak{h}_m(\tau_1)}^{\mathrm{ver}}, \mathcal{O}_{\mathfrak{h}_m(\tau_2)}^{\mathrm{ver}})\big) &= -(p+1)\frac{p^{\mathrm{ord}_p(n)} - 1}{p - 1} \\
\sum_{\ell \geq 0} \chi\big(\mathrm{Tor}_\ell^{\mathcal{O}_{\mathfrak{h}_m}}(\mathcal{O}_{\mathfrak{h}_m(\tau_1)}^{\mathrm{ver}}, \mathcal{O}_{\mathfrak{h}_m(\tau_2)}^{\mathrm{hor}})\big) &= \mathrm{ord}_p(4\overline{Q}_0(\tau_1)) \\
\sum_{\ell \geq 0} \chi\big(\mathrm{Tor}_\ell^{\mathcal{O}_{\mathfrak{h}_m}}(\mathcal{O}_{\mathfrak{h}_m(\tau_1)}^{\mathrm{hor}}, \mathcal{O}_{\mathfrak{h}_m(\tau_2)}^{\mathrm{ver}})\big) &= \mathrm{ord}_p(4\overline{Q}_0(\tau_2)).
\end{aligned}
$$

Combining this with Lemma 7.4 and using

$$\mathrm{ord}_p(4\overline{Q}_0(\tau_1)) + \mathrm{ord}_p(4\overline{Q}_0(\tau_2)) = \mathrm{ord}_p(16\alpha\alpha^\sigma)$$

shows that

$$I_{\mathcal{O}_{Z(T)}}([\mathfrak{D}_Y]^{\mathrm{ver}}, \mathcal{O}_{M_0})$$

$$(7.18) \qquad = 2 \cdot \left(-(p+1)\frac{p^{\mathrm{ord}_p(n)-1}}{p-1} + \mathrm{ord}_p(16\alpha\alpha^\sigma) \right) \cdot |K^\flat\backslash\Omega_0(\tau)\overline{U}^{\mathrm{max},p}/\overline{U}^p|.$$

It is easy to see that

$$|K^\flat\backslash\Omega_0(\tau)\overline{U}^{\mathrm{max},p}/\overline{U}^p| = \frac{|H|}{[\overline{U}_0^{\mathrm{max},p} : \overline{U}_0^p]} \cdot |K^\flat\backslash\Omega_0(s_0)/\overline{U}_0^p|,$$

and the right hand side of this equality is computed in [19, Lemma 11.4]. Combining that calculation with [19, Proposition 9.1] gives

$$2 \cdot |K^\flat\backslash\Omega_0(\tau)\overline{U}^{\mathrm{max},p}/\overline{U}^p| \quad = \quad |H| \cdot \delta(d_K, \mathrm{disc}(B_0)) \cdot H_0(t; \mathrm{disc}(B_0))$$

$$(7.19) \qquad\qquad\qquad\qquad = \quad \frac{|H|}{2} \cdot \deg_{\mathbb{Q}}(\mathcal{Z}(t))$$

where the functions δ and H_0 appearing are those of [19, §8]. Finally, [20, Lemma 7.9.1] tells us that

$$(7.20) \qquad \frac{h_{\widehat{\omega}_0}(\mathcal{Z}^{\mathrm{ver}}(t)_p)}{\log(p)} = -\deg_{\mathbb{Q}}(\mathcal{Z}(t)) \cdot \left(\mathrm{ord}_p(n) - \frac{(p+1)(p^{\mathrm{ord}_p(n)}-1)}{2(p-1)} \right).$$

Combining (7.18), (7.19), and (7.20) with $4t = n^2 d_K$ completes the proof in the case of p inert in K.

If p is ramified or split in K the claim similarly follows from calculations of Kudla-Rapoport-Yang. If p is ramified in K then, as in [19, (11.8)], rewrite (7.16) as

$$\widehat{Z}(T)_{/W} \cong \mathfrak{h}_0(\tau) \times \left(K^\flat\backslash\Omega_0(\tau)\overline{U}^{\mathrm{max},p}/\overline{U}^p \right)$$

and the proof proceeds in exactly the same way as the inert case, by combining the proof of [20, Proposition 7.6.4] with [20, Lemma 7.9.1]. If p is split in K, let $K^{\flat\flat}$ denote the subgroup of elements of $K^\times$ whose image in $(K \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times$ lies in $(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$ and fix an $\epsilon \in K^\times$ whose image in $K \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \mathbb{Q}_p \times \mathbb{Q}_p$ has valuation $(1, -1)$. As in [19, (11.19)], rewrite (7.16) as

$$\widehat{Z}(T)_{/W} \cong \left(\epsilon^{\mathbb{Z}}\backslash\mathfrak{h}_0(\tau) \right) \times \left(K^{\flat\flat}\backslash\Omega_0(\tau)\overline{U}^{\mathrm{max},p}/\overline{U}^p \right).$$

Once again, comparing the proof of [20, Proposition 7.6.4] with [20, Lemma 7.9.1] we find that

$$\frac{1}{|H|} I_{\mathcal{O}_{Z(T)}}([\mathfrak{D}_Y]^{\mathrm{ver}}, \mathcal{O}_{M_0}) + \frac{h_{\widehat{\omega}_0}(\mathcal{Z}^{\mathrm{ver}}(t)_p)}{\log(p)} = 0$$

while [20, Proposition 3.4.5] tells us that $\mathcal{Z}(t)_{/\mathbb{Q}} = \emptyset$.   □

We now construct some cycles on M. As in (6.13) define a horizontal cycle

$$C_p^\bullet = \sum_{D \in \Pi^\bullet(Y)} \mathrm{mult}_D(\mathcal{O}_Y) \cdot \phi(D)$$

of codimension two on M, and define $C_p^{\bullet\bullet}$ in the same way. These cycles are H-invariant and so arise as the pullbacks of horizontal cycles on $\mathcal{M}_{/\mathbb{Z}_p}$, which we

denote by $\mathcal{C}_p^\bullet$ and $\mathcal{C}_p^{\bullet\bullet}$. Now consider the class $[\mathfrak{O}_Y]^{\mathrm{ver}} \in \mathbf{K}_0^{\mathrm{ver}}(Y)$ defined after (7.6). By [12, Proposition 4.2.3] this class lies in the kernel of

$$\mathbf{K}_0^{\mathrm{ver}}(Y) \to \mathbf{K}_0(Y) \to \mathbf{K}_0(\mathrm{Spec}(\mathcal{O}_{Y,\eta}))$$

for every $\eta \in Y$ with $\dim\overline{\{\eta\}} > 1$. Using the notation of [12, §2.2], [12, Lemma 4.2.4] shows that $R\phi_*[\mathfrak{O}_Y]^{\mathrm{ver}} \in F^2\mathbf{K}_0^Y(M)$, while the Gillet-Soulé isomorphism [12, (9)] and the homomorphism [12, (7)] provide us with maps

$$F^2\mathbf{K}_0^Y(M) \to \mathrm{CH}_Y^2(M) \to \mathrm{CH}_{\mathrm{ver}}^2(M).$$

The Chow groups here, as throughout [12], are Chow groups with rational coefficients. The image of $R\phi_*[\mathfrak{O}_Y]^{\mathrm{ver}}$ under this composition, denoted C_p^{ver}, is H-invariant and so arises from some

$$\mathcal{C}_p^{\mathrm{ver}} \in \mathrm{CH}_{\mathrm{ver}}^2(\mathcal{M}_{/\mathbb{Z}_p}).$$

Thus we have constructed a codimension two vertical cycle on $\mathcal{M}_{/\mathbb{Z}_p}$ with rational coefficients, which is determined up the the addition of rational multiples of principal Weil divisors on $\mathcal{M}_{/\mathbb{F}_p}$.

Proposition 7.6. *We have*

$$I_p(\mathcal{C}_p^\bullet, \mathcal{M}_0) + I_p(\mathcal{C}_p^{\mathrm{ver}}, \mathcal{M}_0) + \sum_{\substack{T \in \Sigma(\alpha) \\ \det(T)=0}} \frac{h_{\widehat{\omega}_0}(\mathcal{Z}^{\mathrm{ver}}(t)_p)}{\log(p)}$$

$$= \sum_{\substack{T \in \Sigma(\alpha) \\ \det(T)\neq 0}} e_p(T) \cdot |\overline{\Gamma}_0 \backslash \overline{L}_0(T)| + \frac{1}{2} \cdot \sum_{\substack{T \in \Sigma(\alpha) \\ \det(T)=0}} \deg_\mathbb{Q}(\mathcal{Z}(t)) \cdot \mathrm{ord}_p\left(\frac{4\alpha\alpha^\sigma}{t}\right).$$

Proof. Using the decomposition (2.2) we find

$$I_p(\mathcal{C}_p^\bullet, \mathcal{M}_0) + I_p(\mathcal{C}_p^{\mathrm{ver}}, \mathcal{M}_0)$$

$$= \frac{1}{|H|} \sum_{T \in \Sigma(\alpha)} I_{\mathcal{O}_{Z(T)}}([\mathcal{O}_Y]^\bullet, \mathcal{O}_{M_0}) + \frac{1}{|H|} \sum_{T \in \Sigma(\alpha)} I_{\mathcal{O}_{Z(T)}}([\mathfrak{O}_Y]^{\mathrm{ver}}, \mathcal{O}_{M_0}).$$

The claim now follows from Propositions 7.2, 7.3, and 7.5. $\qquad\square$

8. Pullbacks of arithmetic cycles

We are now ready to put everything together to prove the main result of the paper, Theorem 8.2 below.

Fix an $\alpha \in \mathcal{O}_F$ and a $v \in F \otimes_\mathbb{Q} \mathbb{R}$, both totally positive. Let $\mathcal{C}^{\mathrm{hor}}$, a codimension two cycle on $\mathcal{M}$, be the Zariski closure of the cycle $\mathcal{C}_\mathbb{Q}$ defined by (3.1), and similarly let $\mathcal{C}^\bullet$ and $\mathcal{C}^{\bullet\bullet}$ be the Zariski closures of $\mathcal{C}_\mathbb{Q}^\bullet$ and $\mathcal{C}_\mathbb{Q}^{\bullet\bullet}$. Recall that Proposition 3.2 provides us with Green currents $\Xi^\bullet(\alpha, v)$ and $\Xi^{\bullet\bullet}(\alpha, v)$ for $\mathcal{C}^\bullet$ and $\mathcal{C}^{\bullet\bullet}$, and hence

$$\Xi(\alpha, v) = \Xi^\bullet(\alpha, v) + \Xi^{\bullet\bullet}(\alpha, v)$$

is a Green current for $\mathcal{C}^{\mathrm{hor}}$. Denote by

$$\widehat{\mathcal{Y}}^{\mathrm{hor}}(\alpha, v) \in \widehat{\mathrm{CH}}^2(\mathcal{M})$$

the arithmetic cycle class represented by $(\mathcal{C}^{\mathrm{hor}}, \Xi(\alpha, v))$. We then have a decomposition

$$\widehat{\mathcal{Y}}^{\mathrm{hor}}(\alpha, v) = \widehat{\mathcal{Y}}^\bullet(\alpha, v) + \widehat{\mathcal{Y}}^{\bullet\bullet}(\alpha, v)$$

in which $\widehat{\mathcal{Y}}^\bullet(\alpha, v)$ is the arithmetic cycle class represented by the pair $(\mathcal{C}^\bullet, \Xi^\bullet(\alpha, v))$, and similarly for $\widehat{\mathcal{Y}}^{\bullet\bullet}(\alpha, v)$. For every prime p we have constructed a vertical cycle $\mathcal{C}_p^{\mathrm{ver}}$ of codimension two on $\mathcal{M}$. If $p \nmid \mathrm{disc}(B_0)$ then $\mathcal{C}_p^{\mathrm{ver}}$ was defined at the end of §6, and is nontrivial only if $\mathcal{Y}_0$ has an irreducible component supported in characteristic p. If $p \mid \mathrm{disc}(B_0)$ then $\mathcal{C}_p^{\mathrm{ver}}$ was constructed in §7. In this latter case $\mathcal{C}_p^{\mathrm{ver}}$ has rational coefficients and is only defined up to the addition of rational multiples of principal Weil divisors on $\mathcal{M}_{/\mathbb{F}_p}$. In either case, we endow the cycle $\mathcal{C}_p^{\mathrm{ver}}$ with the trivial Green current to obtain a class

$$\widehat{\mathcal{Y}}_p^{\mathrm{ver}}(\alpha) \in \widehat{\mathrm{CH}}^2(\mathcal{M}).$$

The arithmetic cycle class

$$(8.1) \qquad \widehat{\mathcal{Y}}(\alpha, v) = \widehat{\mathcal{Y}}^{\mathrm{hor}}(\alpha, v) + \sum_{p \text{ prime}} \widehat{\mathcal{Y}}_p^{\mathrm{ver}}(\alpha)$$

agrees with that constructed in [12, §5.1].

Proposition 8.1. *If we abbreviate*

$$b(\alpha, v) = \log\left(\frac{\alpha v_1 + \alpha^\sigma v_2}{4 v_1 v_2 \alpha \alpha^\sigma d_F \mathrm{disc}(B_0)}\right) - J(4\pi\alpha v_1 + 4\pi\alpha^\sigma v_2)$$

(the function J was defined in §5) then

$$
\begin{aligned}
\widehat{\deg}_{\mathcal{M}_0} \widehat{\mathcal{Y}}(\alpha, v) \;=\;& \frac{1}{2} \cdot b(\alpha, v) \cdot \deg_{\mathbb{Q}}(\mathcal{C}^{\bullet\bullet}) - h_{\widehat{\omega}_0}(\mathcal{C}^{\bullet\bullet}) \\
&+ \sum_{\substack{\tau \in \Gamma_0 \backslash L^{\mathrm{nsing}} \\ Q(\tau) = \alpha}} \frac{1}{2 \cdot |\mathrm{Stab}_{\Gamma_0}(\tau)|} \int_{X_0} \xi_0(v_1^{1/2}\tau_1) * \xi_0(v_2^{1/2}\tau_2) \\
&+ \sum_{p \text{ prime}} \log(p)\big(I_p(\mathcal{C}^\bullet, \mathcal{M}_0) + I_p(\mathcal{C}_p^{\mathrm{ver}}, \mathcal{M}_0)\big).
\end{aligned}
$$

Here $\deg_{\mathbb{Q}}$ is defined by (5.11) for irreducible cycles of codimension two on $\mathcal{M}$ and extended linearly to all cycles of codimension two, $\widehat{\omega}_0$ is the metrized Hodge bundle of §4, $h_{\widehat{\omega}_0}$ is the Arakelov height of §5, $\mathrm{Stab}_{\Gamma_0}(\tau)$ is the stabilizer of τ in Γ_0, and L^{nsing} and ξ_0 are as defined in §3.

Proof. From the definition of $\widehat{\mathcal{Y}}(\alpha, v)$, we have

$$(8.2) \qquad \widehat{\deg}_{\mathcal{M}_0} \widehat{\mathcal{Y}}(\alpha, v) = \widehat{\deg}_{\mathcal{M}_0} \Big[\widehat{\mathcal{Y}}^\bullet(\alpha, v) + \sum_p \widehat{\mathcal{Y}}_p^{\mathrm{ver}}(\alpha)\Big] + \widehat{\deg}_{\mathcal{M}_0} \widehat{\mathcal{Y}}^{\bullet\bullet}(\alpha, v).$$

From §2 we know that

$$
\begin{aligned}
\widehat{\deg}_{\mathcal{M}_0} \Big[\widehat{\mathcal{Y}}^\bullet(\alpha, v) + \sum_p \widehat{\mathcal{Y}}_p^{\mathrm{ver}}(\alpha)\Big] \;=\;& \sum_p \log(p)\big[I_p(\mathcal{C}^\bullet, \mathcal{M}_0) + I_p(\mathcal{C}_p^{\mathrm{ver}}, \mathcal{M}_0)\big] \\
(8.3) \qquad\qquad & + I_\infty(\Xi^\bullet(\alpha, v), \mathcal{M}_0),
\end{aligned}
$$

and, as in [12, Proposition 3.2.1],

$$I_\infty(\Xi^\bullet(\alpha, v), \mathcal{M}_0) \;=\; \sum_{\substack{\tau \in \Gamma_0 \backslash L^\bullet \\ Q(\tau) = \alpha}} \frac{1}{|\mathrm{Stab}_{\Gamma_0}(\tau)|} \int_{X_0} \xi_0(v_1^{1/2}\tau_1) * \xi_0(v_2^{1/2}\tau_2)$$

This gives a formula for the first term on the right hand side of (8.2), and we use the adjunction formula of §5 to compute the second term. Indeed, If we extend the

construction $\mathcal{D} \mapsto \widehat{\mathcal{D}}(v)$ of (5.1) linearly to all horizontal cycles $\mathcal{D}$ of codimension two on $\mathcal{M}$ then

$$\widehat{\mathcal{Y}}^{\bullet\bullet}(\alpha, v) = \widehat{\mathcal{C}}^{\bullet\bullet}(\alpha v).$$

Theorem 5.6 extends linearly to all such $\mathcal{D}$ (providing one counts points $P \in \mathcal{D}(\mathbb{C})$ with appropriate multiplicities) and yields

$$\begin{aligned}
\widehat{\deg}_{\mathcal{M}_0} \widehat{\mathcal{Y}}^{\bullet\bullet}(\alpha, v) &= -h_{\widehat{\omega}_0}(\mathcal{C}^{\bullet\bullet}) + \frac{1}{2} b(\alpha, v) \deg_{\mathbb{Q}}(\mathcal{C}^{\bullet\bullet}) \\
&\quad + \frac{1}{2} \sum_{P \in \mathcal{C}^{\bullet\bullet}(\mathbb{C})} e_P^{-1} \sum_{\substack{\gamma \in \Gamma_0 \backslash \Gamma \\ \gamma \notin \Gamma_0}} \int_{X_0} \mathbf{g}_0(\gamma_1 x_0, \alpha_1 v_1) * \mathbf{g}_0(\gamma_2 x_0, \alpha_2 v_2)
\end{aligned}$$

where $x_0 \in X_0$ lies above P under the orbifold uniformization $[\Gamma_0 \backslash X_0] \cong \mathcal{M}_0(\mathbb{C})$. As in the proof of Proposition 3.2, the cycle $\mathcal{C}^{\bullet\bullet}(\mathbb{C})$ on $\mathcal{M}_0(\mathbb{C})$ is identified with the formal sum

$$\mathcal{C}^{\bullet\bullet}(\mathbb{C}) = \sum_{\substack{\tau \in \Gamma \backslash L^{\bullet\bullet} \\ Q(\tau) = \alpha}} (x^+(\tau) + x^-(\tau)).$$

Choosing each coset representative $\tau \in \Gamma \backslash L^{\bullet\bullet}$ to lie in L^{sing}, so that $x^{\pm}(\tau) \in X_0$, we see from Lemma 5.1 and (3.4) that

$$\gamma\tau \in L^{\mathrm{sing}} \iff \gamma x^{\pm}(\tau) \in X_0 \iff \gamma \in \Gamma_0.$$

This observation gives the second equality of

$$\begin{aligned}
\sum_{P \in \mathcal{C}^{\bullet\bullet}(\mathbb{C})} & e_P^{-1} \sum_{\substack{\gamma \in \Gamma_0 \backslash \Gamma \\ \gamma \notin \Gamma_0}} \int_{X_0} \mathbf{g}_0(\gamma_1 x_0, \alpha_1 v_1) * \mathbf{g}_0(\gamma_2 x_0, \alpha_2 v_2) \\
&= \sum_{\substack{\tau \in \Gamma \backslash L^{\bullet\bullet} \\ Q(\tau) = \alpha}} \frac{1}{|\mathrm{Stab}_{\Gamma_0}(\tau)|} \sum_{\substack{\gamma \in \Gamma_0 \backslash \Gamma \\ \gamma \notin \Gamma_0}} \int_{X_0} \xi_0(v_1^{1/2} \tau_1) * \xi_0(v_2^{1/2} \tau_2) \\
&= \sum_{\substack{\tau \in \Gamma_0 \backslash (L^{\bullet\bullet} \smallsetminus L^{\mathrm{sing}}) \\ Q(\tau) = \alpha}} \frac{1}{|\mathrm{Stab}_{\Gamma_0}(\tau)|} \int_{X_0} \xi_0(v_1^{1/2} \tau_1) * \xi_0(v_2^{1/2} \tau_2),
\end{aligned}$$

and using $L^{\mathrm{nsing}} = L^{\bullet} \sqcup (L^{\bullet\bullet} \smallsetminus L^{\mathrm{sing}})$ we obtain

$$\begin{aligned}
I_{\infty}(\Xi^{\bullet}(\alpha, v), \mathcal{M}_0) + \widehat{\deg}_{\mathcal{M}_0} \widehat{\mathcal{Y}}^{\bullet\bullet}(\alpha, v) &= -h_{\widehat{\omega}_0}(\mathcal{C}^{\bullet\bullet}) + \frac{1}{2} b(\alpha, v) \deg_{\mathbb{Q}}(\mathcal{C}^{\bullet\bullet}) \\
&\quad + \sum_{\substack{\tau \in \Gamma_0 \backslash L^{\mathrm{nsing}} \\ Q(\tau) = \alpha}} \frac{1}{2|\mathrm{Stab}_{\Gamma_0}(\tau)|} \int_{X_0} \xi_0(v_1^{1/2} \tau_1) * \xi_0(v_2^{1/2} \tau_2).
\end{aligned}$$

Combining this last equality with (8.2) and (8.3) completes the proof. $\qquad \square$

For every symmetric positive definite matrix $\mathbf{v} \in M_2(\mathbb{R})$, and every $T \in \mathrm{Sym}_2(\mathbb{Z})^{\vee}$, Kudla-Rapoport-Yang [20] have defined an arithmetic cycle class

$$(8.4) \qquad\qquad \widehat{\mathcal{Z}}(T, \mathbf{v}) \in \widehat{\mathrm{CH}}_{\mathbb{R}}^2(\mathcal{M}_0)$$

in the $\mathbb{R}$-arithmetic Chow group defined in [20, §2.4]. Our main result, an arithmetic form of the decomposition (2.2), relates the arithmetic degree of (8.1) along $\mathcal{M}_0$, in the sense of (2.8), with the arithmetic degree of (8.4), in the sense of [20, (2.4.10)].

Recall that (v_1, v_2) denotes the image of v under $F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R} \times \mathbb{R}$ and that $\{\varpi_1, \varpi_2\}$ is our fixed $\mathbb{Z}$-basis of $\mathcal{O}_F$. Define

$$(8.5) \qquad \mathbf{v} = R \begin{pmatrix} v_1 & \\ & v_2 \end{pmatrix} {}^t R \qquad R = \begin{pmatrix} \varpi_1 & \varpi_1^\sigma \\ \varpi_2 & \varpi_2^\sigma \end{pmatrix}.$$

Theorem 8.2. *If either*

(a) $F(\sqrt{-\alpha})/\mathbb{Q}$ *is not biquadratic, or*
(b) 2 *splits in F and* $\gcd(\alpha \mathcal{O}_F, \mathfrak{D}_F) = \mathcal{O}_F$

then

$$\widehat{\deg}_{\mathcal{M}_0} \widehat{\mathcal{Y}}(\alpha, v) = \sum_{T \in \Sigma(\alpha)} \widehat{\deg} \, \widehat{\mathcal{Z}}(T, \mathbf{v}).$$

Proof. By [20, Theorem 3.6.1], $\mathcal{Z}(T)_{/\mathbb{Q}} = \emptyset$ for $\det(T) \neq 0$. Passing to the generic fiber in (2.2) yields an isomorphism of stacks

$$\bigsqcup_{\substack{T \in \Sigma(\alpha) \\ \det(T) = 0}} \mathcal{Z}(T)_{/\mathbb{Q}} \cong \mathcal{Y}_0(\alpha)_{/\mathbb{Q}},$$

which, after applying (6.8) and taking Zariski closures, gives the equality of cycles

$$\sum_{\substack{T \in \Sigma(\alpha) \\ \det(T) = 0}} \mathcal{Z}^{\mathrm{hor}}(t) = \mathcal{C}^{\bullet\bullet}$$

of codimension one in $\mathcal{M}_0$, where t is the positive integer defined by (6.7). As in the proof of [20, Lemma 7.9.1] (*i.e.* combining [20, (6.4.2)], [19, (9.12)], [19, Proposition 12.1], and [19, Proposition 9.1]) and using $\mathrm{Tr}(T\mathbf{v}) = \alpha v_1 + \alpha^\sigma v_2$, we have

$$\widehat{\deg} \, \widehat{\mathcal{Z}}(T, \mathbf{v}) = -h_{\widehat{\omega}_0}(\mathcal{Z}^{\mathrm{hor}}(t)) - h_{\widehat{\omega}_0}(\mathcal{Z}^{\mathrm{ver}}(t))$$
$$+ \frac{1}{2} \deg_{\mathbb{Q}}(\mathcal{Z}(t)) \cdot \left[\log\left(\frac{\alpha v_1 + \alpha^\sigma v_2}{t v_1 v_2 d_F \mathrm{disc}(B_0)} \right) - J(4\pi \alpha v_1 + 4\pi \alpha^\sigma v_2) \right]$$

for every singular $T \in \Sigma(\alpha)$. From this we deduce

$$\sum_{\substack{T \in \Sigma(\alpha) \\ \det(T) = 0}} \widehat{\deg} \, \widehat{\mathcal{Z}}(T, \mathbf{v}) = -h_{\widehat{\omega}_0}(\mathcal{C}^{\bullet\bullet}) - \sum_{\substack{T \in \Sigma(\alpha) \\ \det(T) = 0}} h_{\widehat{\omega}_0}(\mathcal{Z}^{\mathrm{ver}}(t))$$
$$+ \frac{1}{2} \deg_{\mathbb{Q}}(\mathcal{C}^{\bullet\bullet}) \cdot b(\alpha, v) + \frac{1}{2} \sum_{\substack{T \in \Sigma(\alpha) \\ \det(T) = 0}} \deg_{\mathbb{Q}}(\mathcal{Z}(t)) \cdot \log\left(\frac{4\alpha\alpha^\sigma}{t} \right).$$

Recall from §3 that V is the F-vector space of trace zero elements of B, and V_0 is the $\mathbb{Q}$-vector space of trace zero elements of B_0. Let $L_0 \subset V_0$ be the $\mathbb{Z}$-submodule of trace zero elements of $\mathcal{O}_{B_0}$, with Γ_0 acting on L_0 by conjugation. For each $T \in \mathrm{Sym}_2(\mathbb{Z})^\vee$ let $L_0(T) \subset L_0 \times L_0$ be the subset of pairs (s_1, s_2) satisfying (2.1), where $[s_1, s_2] = -\mathrm{Tr}(s_1 s_2)$. For each nonsingular $T \in \Sigma(\alpha)$ with $\mathrm{Diff}(T, B_0) = \{\infty\}$ (in the sense of [20, §3.6]) and $(s_1, s_2) \in L_0(T)$, define

$$\tau = s_1 \varpi_2 + s_2 \varpi_2 \in V \cong V_0 \otimes_{\mathbb{Q}} F,$$

and let (τ_1, τ_2) be the image of τ under

$$V \otimes_{\mathbb{Q}} \mathbb{R} \cong V_0 \otimes_{\mathbb{Q}} \mathbb{R} \times V_0 \otimes_{\mathbb{Q}} \mathbb{R}.$$

Then, by [20, §6.3],

$$\widehat{\deg}\,\widehat{\mathcal{Z}}(T,\mathbf{v}) = \sum_{(s_1,s_2)\in\Gamma_0\backslash L_0(T)} \frac{1}{2\cdot e_{s_1,s_2}} \int_{X_0} \xi_0(v_1^{1/2}\tau_1) * \xi_0(v_2^{1/2}\tau_2),$$

where we have abbreviated $e_{s_1,s_2} = |\mathrm{Stab}_{\Gamma_0}(s_1)\cap\mathrm{Stab}_{\Gamma_0}(s_2)|$. Using the bijection

$$\bigsqcup_{\substack{T\in\Sigma(\alpha)\\ \det(T)\neq 0}} L_0(T) \to \{\tau\in L^{\mathrm{nsing}} : Q(\tau)=\alpha\}$$

defined by $(s_1,s_2)\mapsto s_1\varpi_1 + s_2\varpi_2$ and the fact that $L(T)=\emptyset$ unless $\mathrm{Diff}(T,B_0) = \{\infty\}$, we obtain

$$\sum_{\substack{T\in\Sigma(\alpha)\\ \det(T)\neq 0\\ \mathrm{Diff}(T,B_0)=\infty}} \widehat{\deg}\,\widehat{\mathcal{Z}}(T,\mathbf{v}) = \sum_{\substack{\tau\in\Gamma_0\backslash L^{\mathrm{nsing}}\\ Q(\tau)=\alpha}} \frac{1}{2\cdot|\mathrm{Stab}_{\Gamma_0}(\tau)|} \int_{X_0} \xi_0(v_1^{1/2}\tau_1) * \xi_0(v_2^{1/2}\tau_2).$$

Using Proposition 8.1 and the above formulas, we are reduced to verifying

$$\sum_{\substack{T\in\Sigma(\alpha)\\ \det(T)=0}} h_{\widehat{\omega}_0}(\mathcal{Z}^{\mathrm{ver}}(t)) + \sum_{p\text{ prime}} \left[I_p(\mathcal{C}^\bullet,\mathcal{M}_0) + I_p(\mathcal{C}_p^{\mathrm{ver}},\mathcal{M}_0)\right]\cdot\log(p)$$

$$(8.6)\quad = \sum_{p<\infty}\sum_{\substack{T\in\Sigma(\alpha)\\ \det(T)\neq 0\\ \mathrm{Diff}(B_0,T)=\{p\}}} \widehat{\deg}\,\widehat{\mathcal{Z}}(T,\mathbf{v}) + \frac{1}{2}\sum_{\substack{T\in\Sigma(\alpha)\\ \det(T)=0}} \deg_{\mathbb{Q}}(\mathcal{Z}(t))\cdot\log\left(\frac{4\alpha\alpha^\sigma}{t}\right).$$

For a prime p that does not divide $\mathrm{disc}(B_0)$ and a nonsingular $T\in\Sigma(\alpha)$ with $\mathrm{Diff}(B_0,T) = \{p\}$, the arithmetic cycle class $\widehat{\mathcal{Z}}(T,\mathbf{v})$ is studied in [20, §6.1]. Comparing with Proposition 6.7 gives

$$\left[I_p(\mathcal{C}_p^\bullet,\mathcal{M}_0) + I_p(\mathcal{C}_p^{\mathrm{ver}},\mathcal{M}_0)\right]\cdot\log(p)$$

$$= \sum_{\substack{T\in\Sigma(\alpha)\\ \det(T)\neq 0\\ \mathrm{Diff}(T,B_0)=\{p\}}} \widehat{\deg}\,\widehat{\mathcal{Z}}(T,\mathbf{v}) + \frac{1}{2}\sum_{\substack{T\in\Sigma(\alpha)\\ \det(T)=0}} \deg_{\mathbb{Q}}(\mathcal{Z}(t))\cdot\mathrm{ord}_p\left(\frac{4\alpha\alpha^\sigma}{t}\right)\log(p).$$

For a prime p dividing $\mathrm{disc}(B_0)$ and a nonsingular $T\in\Sigma(\alpha)$ with $\mathrm{Diff}(B_0,T) = \{p\}$, the arithmetic cycle class $\widehat{\mathcal{Z}}(T,\mathbf{v})$ is studied in [20, §6.2]. Comparing with Proposition 7.6 gives

$$\left[I_p(\mathcal{C}_p^\bullet,\mathcal{M}_0) + I_p(\mathcal{C}_p^{\mathrm{ver}},\mathcal{M}_0)\right]\cdot\log(p) + \sum_{\substack{T\in\Sigma(\alpha)\\ \det(T)=0}} h_{\widehat{\omega}_0}(\mathcal{Z}^{\mathrm{ver}}(t)_p)$$

$$= \sum_{\substack{T\in\Sigma(\alpha)\\ \det(T)\neq 0\\ \mathrm{Diff}(T,B_0)=\{p\}}} \widehat{\deg}\,\widehat{\mathcal{Z}}(T,\mathbf{v}) + \frac{1}{2}\sum_{\substack{T\in\Sigma(\alpha)\\ \det(T)=0}} \deg_{\mathbb{Q}}(\mathcal{Z}(t))\cdot\mathrm{ord}_p\left(\frac{4\alpha\alpha^\sigma}{t}\right)\log(p).$$

The above two formulas prove (8.6), and complete the proof of the theorem.   □

Corollary 8.3. *Suppose that $\alpha \in \mathcal{O}_F$ and $v \in F \otimes_{\mathbb{Q}} \mathbb{R}$ are both totally positive. If either $F(\sqrt{-\alpha})$ is not biquadratic, or if 2 splits in F and $\alpha \mathcal{O}_F$ is relatively prime to the different of $F/\mathbb{Q}$, then*

$$\widehat{\deg}_{\mathcal{M}_0} \widehat{\mathcal{Y}}(\alpha, v) = c(\alpha, v)$$

where $c(\alpha, v)$ is the Fourier coefficient appearing in (1.4).

Proof. By [12, Lemma 5.2.1]

$$c(\alpha, v) = \sum_{T \in \Sigma(\alpha)} \widehat{\deg}\, \widehat{\mathcal{Z}}(T, \mathbf{v})$$

and so the proof is immediate from Theorem 8.2. $\qquad\square$

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